# Math 228: Homework 5

Mary Radcliffe

due 19 Oct 2016

Complete the following problems. Fully justify each response. You need only turn in those problems marked with a (*).

1. (*) Let $\pi$ be a permutation of $[n]$. We define a cycle of length $k$ in $\pi$ to be a sequence $a_1, a_2, \ldots, a_k$ such that $\pi(a_i) = a_{i+1}$ for all $i < k$, and $\pi(a_k) = a_1$. Note that every permutation can be written as a collection of cycles.

   (a) Fix $k$. Find the exponential generating function $F(x)$ for the number of permutations having all cycles of length $k$.

   (b) Determine the number of permutations of $[n]$ that have all cycles of length at most 4.

2. (*) Suppose you wanted to count permutations that had all their cycles of length divisible by $k$. Define an exponential family $\mathcal{F}$ that will generate these permutations. Use the theorems from class to find an exponential generating function for the number of such permutations.

3. Suppose you have an exponential family $\mathcal{F} = \{D_1, D_2, D_3, \ldots\}$, with corresponding generating functions

$$D(x) = \sum_{n\geq 1} \frac{d_n}{n!} x^n \text{ and } H(x) = \sum_{n\geq 0} \frac{h_n}{n!} x^n,$$

   where $d_n = |D_n|$ and $h_n$ is the total number of hands of weight $n$ in $\mathcal{F}$.

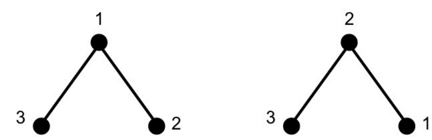
   For $n \geq 1$, put $h_n = \sum_{k=1}^{n} h(n,k)$. Prove that $nh_n = \sum_{k=1}^{n-1} \binom{n}{k} k d_k h_{n-k}$.

   (Hint: Use the corollary from class to write $H(x)$ in terms of $D(x)$. Take a logarithm of both sides, then take a derivative and solve for the desired coefficients.)

4. Let $G$ be a graph. Show that $G$ must have at least two vertices of the same degree.

5. (*) Prove that the distances in a graph $G$ satisfy the triangle inequality; that is, if $u, v, w \in V(G)$ then $d(u,w) \leq d(u,v) + d(v,w)$.

6. (*) A labelled graph is a graph together with a labeling on the vertices. For example, the two graphs shown below are the same as unlabelled graphs, but are different as labelled graphs:




(a) Explain why there are $2^{\binom{n}{2}}$ labelled graphs on $n$ vertices.

(b) Use the result of problem **??** to find a recurrence for calculating the number of connected labelled graphs on $n$ vertices.

(c) Determine the number of connected labelled graphs on $n$ vertices for $n = 1, 2, 3, 4, 5$.