Math 127 Homework

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Complete the following problems. Fully justify each response. You need only turn in those problems marked with a (*).

1. (*) Use Euler's Totient Theorem to calculate $2^{105723} \pmod{13}$.
2. Let $n \in \mathbb{N}$ and let $a \in \mathbb{Z}$ with $a \perp n$. Euler's Totient Theorem tells us that $a^{\varphi(n)} \equiv 1 \pmod{n}$. Is it true that the order of a must be $\varphi(n)$? If so, prove it. If not, explain why not.
3. (*) Let $a, b \in \mathbb{N}$, with $a \perp b$. Prove that $\varphi(ab) = \varphi(a)\varphi(b)$.

   (Hint: Define $S_{ab} = \{m \in \mathbb{N} \mid m < ab, m \perp ab\}$, $S_a = \{m \in \mathbb{N} \mid m < ab, m \perp a\}$, and $S_b = \{m \in \mathbb{N} \mid m < ab, m \perp b\}$. Define a function $f : S_{ab} \to S_a \times S_b$ by $f(x) = (x \pmod{a}, x \pmod{b})$. Show that this is well-defined and bijective.)
4. (*) Let m, n be positive integers with $m \perp n$. Let x, y be integers, such that $x \equiv y \pmod{n}$ and $x \equiv y \pmod{m}$. Prove that $x \equiv y \pmod{mn}$.
5. (*) If possible, solve the following system of congruences. If not possible, explain why not:

$$\begin{aligned} x &\equiv 4 \pmod{11} \\ x &\equiv 3 \pmod{17} \\ x &\equiv 6 \pmod{18} \end{aligned}$$

6. If possible, solve the following system of congruences. If not possible, explain why not:

$$\begin{aligned} 20x &\equiv 9 \pmod{30} \\ 9x &\equiv 12 \pmod{33} \\ 36x &\equiv 48 \pmod{60} \end{aligned}$$

7. (*) A troop of 17 monkeys store their bananas in 11 piles of equal size, each containing more than one banana, with a twelfth pile of 6 left over. When they divide the bananas into 17 equal piles, none are left over. What is the smallest number of bananas they can have?
8. Let $n \in \mathbb{N}$. For each $1 \leq i \leq n$, let $a_i, b_i \in \mathbb{R}$ be real numbers with $a_i < b_i$, and let $I_i = [a_i, b_i]$, the corresponding real interval.

   Define a relation $\leq$ on $\{I_1, I_2, \ldots, I_n\}$ by $I_i \leq I_j$ iff $b_i \leq a_j$ or $i = j$. Prove that this is a partial order.
9. (CHALLENGE: BONUS 10 points) Let $n \in \mathbb{N}$ have prime decomposition $n = p_1^{k_1} p_2^{k_2} \ldots p_r^{k_r}$. Use inclusion-exclusion to write $\varphi(n)$ in terms of the primes $p_1, p_2, \ldots, p_r$.