Math 127 Homework

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Complete the following problems. Fully justify each response. You need only turn in those problems marked with a (*).

For these problems, X, Y, Z always represent sets.

1. (*)
   (a) Prove that there exists an injection $f : X \to Y$ if and only if there exists a surjection $g : Y \to X$.
   (b) Use part (a) to show that there exists a bijection $f : X \to Y$ if and only if there exists a bijection $g : Y \to X$.
2. Let $f : X \to Y$.
   (a) If g is a left inverse to f, is it necessarily injective? surjective? Give a proof of those properties g must possess, and a counterexample to the others.
   (b) If h is a right inverse to f, is it necessarily injective? surjective? Give a proof of those properties h must possess, and a counterexample to the others.
3. (*) Your friend writes the following.

   > Suppose that $f : X \to Y$ is injective, but not surjective. Then there does not exist a bijection between X and Y.

   Give an example to show that your friend is wrong, in general. What condition(s) could you put on the sets X and/or Y to make the statement true?
4. (*) Prove that if $f : X \to Y$ and $g : Y \to Z$ are both bijections, then $g \circ f$ is a bijection, and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.
5. Let $f : X \to Y$ be a function, and let $A \subseteq X$. In general, it is not true that $f^{-1}(f(A)) = A$. Is it true when f is an injection? surjection? bijection?
6. (*) Show that for any finite set X, there exists an injection $i : X \to \mathcal{P}(X)$, but that there does not exist a bijection. Explain why this implies that $|\mathcal{P}(X)| > |X|$.
7. Carefully prove Claims 1 and 2 from the proof of Lemma 1 in the Finite Cardinality Notes.
8. Prove Corollary 3 to Lemma 1 in the Finite Cardinality Notes.
9. (*) Let X and Y be finite sets. Prove that $f : X \to Y$ is injective if and only if $\forall A \subseteq Y, |f^{-1}(A)| \leq |A|$.
10. Let X and Y be finite sets. Prove that $f : X \to Y$ is surjective if and only if $\forall A \subseteq Y, |f^{-1}(A)| \geq |A|$.