Math 127 Homework

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Complete the following problems. Fully justify each response. You need only turn in those problems marked with a (*).

1. Prove the following claim. Explain what proof technique(s) you are using, and why.

   Let n be a positive integer. If n is a perfect square, then n leaves a remainder of 0 or 1 when divided by 4.

2. (*) Use the method of contrapositive to prove the Pigeonhole Principle, as stated below:

   Let m objects be distributed into n bins. If $m > n$, then at least one bin contains at least two objects.

3. (*) Prove that for all real numbers x,

   $$-5 \leq |x+2| - |x-3| \leq 5.$$

4. Prove the following claim. Explain what proof technique(s) you are using, and why.

   Let a, b, c be positive real numbers. If $ab \geq c$, then at least one of a, b is greater than or equal to $\sqrt{c}$.

5. Consider the following Theorem and Proof.

   Theorem 1. *For any positive integer n, the sum of all positive integers less than n is not equal to n.*

   Proof. AFSOC that for any positive integer n, the sum of all positive integers is equal to n. But then consider $n = 5$: notice that $1+2+3+4 = 10 \neq 5$. Therefore, we have reached a contradiction, and hence for any positive integer n, the sum of all positive integers less than n is not equal to n. □

   Is this result correct? Is the proof correct? What mistakes have been made?

6. Prove the following statement in three ways: directly, by contradiction, and by contrapositive. Be very explicit about the different assumptions you make in each of the three proofs.

   Let x, y be real numbers. If $x^2 y = 2x + y$, then if $y \neq 0$, then $x \neq 0$.

7. (*) Prove the following statement in three ways: directly, by contradiction, and by contrapositive. Be very explicit about the different assumptions you make in each of the three proofs.

   Let a, b be real numbers. If $a \leq x$ for every $x > b$, then $a \leq b$.

8. (*) Consider the following statement.

   Let n be a positive integer, and let $y_1, y_2, \ldots, y_n$ be real numbers. Let $A = \frac{y_1+y_2+\cdots+y_n}{n}$ be the average of the y_i's. There is at least one integer i with $1 \leq i \leq n$ such that $y_i \geq A$, and there is at least one integer j with $1 \leq j \leq n$ such that $y_j \leq A$.

Explain why you cannot prove this statement by contrapositive. Then prove it using some other technique.

9. Give an example of a proposition of the form $\forall x, \exists y, p(x, y) \wedge q(x)$. You need not be concerned with the truth value of your proposition. Write the formal negation of your statement.

10. (*) Suppose that x is a member of the positive integers $\{1, 2, 3, \dots\}$ and y is a member of the rationals. Let $p(x, y)$ be the statement that $xy = 1$. Consider the following two statements:

    (a) $\forall x, \exists y, p(x, y)$

    (b) $\exists y, \forall x, p(x, y)$

    How do these statements differ? Are either of them true?