21-127 Final Exam Practice Problems

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1 Logic and Proof Techniques

1. Let p, q be logical propositions. Prove that $(\neg(p \wedge q)) \wedge (p \vee q)$ is logically equivalent to $(p \wedge \neg q) \vee (q \wedge \neg p)$.

2. Prove that $\sqrt{p}$ is irrational for any prime $p > 0$.

3. Find the smallest $n > 0$ such that $n!$ is divisible by 990.

4. Let p and q be logical propositions.

   (a) Prove that $p \Rightarrow q$ is logically equivalent to $\neg q \Rightarrow \neg p$.

   (b) Explain in words why the above statement makes sense.

5. Let p, q, r be logical propositions. Prove that

$$[p \Rightarrow (q \Rightarrow r)] \Rightarrow [(p \Rightarrow q) \Rightarrow (p \Rightarrow r)]$$

   is tautologically true (that is, it is true regardless of the truth values of p, q, r).

2 Induction

1. Use induction to prove that $\sum_{i=1}^{k}(2i-1) = k^2$.

2. Let $n \in \mathbb{N}$ be written, in base 10, as $111\cdots 11$, where there are 3^k 1s in the base expansion. Prove that n is divisible by 3^k.

3. Suppose you draw n straight lines in the plane, where no two lines are parallel and no three lines meet at a point. How many regions have you divided the plane into? Prove that your answer is correct.

4. Define a sequence by $a_0 = 0$, $a_1 = 1$, and $a_n = 3a_{n-1} - 2a_{n-2}$ for $n \geq 2$. Derive and prove a nonrecursive formula for a_n.

5. Prove that for $x_1, x_2, \dots, x_n \in \mathbb{R}$, with $x_i \geq 0$ for all i,

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \dots x_n}.$$

6. Let $k \in \mathbb{N}$, with $k \neq 0$. Prove that $k(k+1)(k+2)$ is divisible by 3.

3 Set Theory and functions

1. Let $U_1, U_2, \ldots, U_k$ be a finite partition of a set X, and let $A \subseteq X$. Prove that $A = \bigcup_{i=1}^{k} (A \cap U_i)$.

2. Let X and Y be sets. Prove that $X \subseteq Y$ if and only if $X = Y \backslash (Y \backslash X)$.

3. Define a function $f : \mathbb{R} \to \mathbb{R}$ by the following: for every $a \in \mathbb{R}$, let $f(a) = x$, where $x^2 + 2ax + a^2 = 0$. Prove that this function is well-defined.

4. Let X and Y be sets, and let $f : X \to Y$ be a function.

   (a) For $A, B \subseteq X$, prove that $f(A \cup B) = f(A) \cup f(B)$.

   (b) For $A, B \subseteq X$, prove that $f(A \cap B) \subseteq f(A) \cap f(B)$, but that sometimes these sets may not be equal.

5. Let $f : X \to Y$ be a function. Prove that f is injective if and only if $f(A \backslash B) = f(A) \backslash f(B)$ for every $A, B \subseteq X$.

6. Let $f : X \to X$ be a function, where X is a finite set. Prove that f is injective if and only if f is surjective. Explain why this is not true in the case that X is infinite.

7. We know that function composition is associative. Is it also commutative? Why/why not?

4 Counting: finite

1. (a) How many ways are there to rearrange the letters in the word "VECTOR"?

   (b) How many ways are there to rearrange the letters in the word "TRUST," in such a way that the two Ts are not next to each other?

   (c) How many ways are there to rearrange the letters in the word "MATHEMATICS" so that no two consecutive letters are the same?

2. Let $n \in \mathbb{N}$, with $n \geq 1$. How many surjective functions are there from $[n+1]$ to $[n]$?

3. Prove that for all $n, m, k \in \mathbb{N}$, we have

$$\sum_{\ell=0}^{k} \binom{n}{\ell} \binom{m}{k-\ell} = \binom{n+m}{k}$$

4. Use counting in two ways to prove that for all $n, k \in \mathbb{N}$ with $n \geq k > 0$, we have

$$\sum_{j=n-k}^{n} \binom{n}{j} = \sum_{j=n-k}^{n} \binom{j-1}{n-k-1} 2^{n-j}.$$

5. Use Inclusion-Exclusion to determine the number of subsets of [20] that contain a multiple of 5.

6. Suppose we have a box containing a set of standard chess pieces. You select three pieces from the box. In how many ways can you select the pieces so that

(a) all three are the same color.

(b) all three are the same color and all three are pawns.

(c) all three are rooks.

7. From a standard deck of cards, you deal out a five-card poker hand. How many different hand have at least three cards of the same type (i.e., a 3-of-a-kind or 4-of-a-kind).

5 Counting: infinite

1. Suppose that A, B, C are countably infinite disjoint sets. Prove that $A \cup B \cup C$ is countably infinite directly, by finding a bijection between $A \cup B \cup C$ and $\mathbb{N}$.

2. Let $\mathcal{F} = \{f : X \to \{0, 1\}\}$, where X is any set. Carefully go through the Cantor diagonalization argument to show that $|\mathcal{F}| > |X|$.

3. Let X, Y be nonempty sets of positive real numbers. Define

$$XY = \{xy \mid x \in X, y \in Y\}.$$

   Prove that XY is infinite if and only if at least one of X or Y is infinite.

4. Let $\mathcal{A}$ be a collection of sets. We say that $\mathcal{A}$ has the *finite intersection property* if the intersection of any finite number of sets in $\mathcal{A}$ is nonempty.

   Give an example of an infinite collection of sets $\mathcal{A}$ that have the finite intersection property, but $\bigcap_{A \in \mathcal{A}} A$ is empty.

6 Divisibility and Number Theory

1. Use the Euclidean Algorithm to prove that for all $n \in \mathbb{N}$, the fraction $\frac{12n+1}{30n+2}$ is in lowest terms.

2. Suppose that $a, b, c, d \in \mathbb{N}$ with $ab - cd$ divides each of a, b, c, and d. Prove that $ab - cd = \pm 1$.

3. Find the set of all integer solutions to the equation $3x + 4y = 5$.

7 Modular arithmetic

1. Suppose that $a \equiv b \pmod{n}$, and $c \equiv d \pmod{n}$. Prove that $ac \equiv bd \pmod{n}$.

2. Calculate the remainder of 3^{1000} when divided by 7.

3. Suppose that $a, b, c \in \mathbb{Z}_n$, and $a + b \equiv a + c \mod n$. Is it true that $b \equiv c \mod n$? If so, prove it. If not, explain why not.

4. Find the set of all solutions to the congruences

$$\begin{aligned} x &\equiv 7 \mod 11 \\ x &\equiv 3 \mod 5 \\ x &\equiv 1 \mod 6 \end{aligned}$$

5. Suppose $n > 1$ is an integer such that $4((n-1)! + 1) \equiv 0 \mod n$. Prove that $n = 4$ or n is prime.

6. Let $S \subseteq [2n]$ with $|S| \geq n+1$. Prove that there exist $a, b \in S$, $a \neq b$, with $a|b$.

8 Posets

1. Define a relation $\preceq$ on $\mathbb{N}$ by $x \preceq y$ if and only if $x \leq y$ and x and y have the same parity. Is $\mathbb{N}$ a poset under $\preceq$? If so, prove it. If not, explain why not.

2. Define a relation $\preceq$ on $\mathbb{N}$ by $x \preceq y$ if and only if $x \leq y$ and $x \perp y$. Is $\mathbb{N}$ a poset under $\preceq$? If so, prove it. If not, explain why not.

9 Sample Short Answer problems:

1. Explain, in your own words, how and why the Principle of Weak Induction works.

2. Explain, in your own words, how and why the Principle of Strong Induction works.

3. When does a function have an inverse? Why?

4. Let $p(x, y)$ be a logical formula. Explain the difference between the meaning of the statements $\forall x, \exists y, p(x, y)$ and $\exists y, \forall x, p(x, y)$, and how the proofs of these statements might also differ.

5. Explain, perhaps using pictures, why two sets are the same size if and only if there is a bijection between them.

6. What is the Inductive Axiom (from the Peano Axioms)? How is it useful?

7. Explain the relationship between multiplicative inverses and gcds. Why does this relationship exist?

8. State the Euler Totient Theorem. Why is this theorem useful?

9. Describe Cantor's diagonalization method, and what it can be used to prove.