21-127 Exam 2 Review Materials Solutions

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1. Let A, B, C be subsets of the universe Z. Prove each of the following identities:

   (a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

   Solution: We prove equality by double containment.
   ($\subseteq$) Let $x \in A \cap (B \cup C)$. Then by definition of intersection, $x \in A$ and $x \in B \cup C$. Since $x \in B \cup C$ we must have either $x \in B$ or $x \in C$. If $x \in B$, then as $x \in A$, we have $x \in A \cap B$. Likewise, if $x \in C$, then $x \in A \cap C$. Therefore, as x is in at least one of $A \cap B$ or $A \cap C$, we have $x \in (A \cap B) \cup (A \cap C)$. Thus, $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

   ($\supseteq$) Let $x \in (A \cap B) \cup (A \cap C)$. Then either $x \in A \cap B$ or $x \in A \cap C$; wlog suppose $x \in A \cap B$. Then $x \in A$ and $x \in B$, by definition of intersection. As $x \in B$, it is also the case that $x \in B \cup C$, and hence $x \in A \cap (B \cup C)$. Therefore, $A \cap (B \cup C) \supseteq (A \cap B) \cup (A \cap C)$.

   As both containments are met, we therefore have equality: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

   (b) $(A \cap B) \cup (A \cap B^c) = A$

   Solution: We prove equality by double containment.
   ($\subseteq$) Let $x \in (A \cap B) \cup (A \cap B^c)$. Then either $x \in A \cap B$ or $x \in A \cap B^c$. In either of these two cases, we must have $x \in A$ by the definition of intersection. Therefore, $(A \cap B) \cup (A \cap B^c) \subseteq A$.

   ($\supseteq$) Let $x \in A$. By the Law of Excluded Middle, either $x \in B$ or $x \in B^c$. Therefore, either $x \in A \cap B$ or $x \in A \cap B^c$. By definition of union, we therefore have $x \in (A \cap B) \cup (A \cap B^c)$. Thus, $(A \cap B) \cup (A \cap B^c) \supseteq A$.

   As both containments are met, we therefore have equality: $(A \cap B) \cup (A \cap B^c) = A$.

   (c) $(A^c \cap Z)^c = A$

   Solution: Note that $A^c \subseteq Z$, and hence $A^c \cap Z = A^c$. Therefore, it is sufficient to prove that $(A^c)^c = A$. Notice,

   $$\begin{aligned} (A^c)^c &= \{x \in Z \mid x \notin A\}^c \\ &= \{x \in Z \mid \neg(x \notin A)\} \\ &= \{x \in Z \mid x \in A\} \\ &= A. \end{aligned}$$

   Therefore, $(A^c)^c = A$, and thus $(A^c \cap Z)^c = A$.

(d) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Solution: We prove equality by double containment.
($\subseteq$) Let $x \in A \times (B \cap C)$. Then we may write x as an ordered pair (a, b), where $a \in A$ and $b \in B \cap C$. Note that as $b \in B \cap C$, we have $b \in B$ and $b \in C$. Therefore, $(a, b) \in A \times B$ and $(a, b) \in A \times C$. Thus, by definition of intersection, $(a, b) \in (A \times B) \cap (A \times C)$, and hence $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$

($\supseteq$) Let $x \in (A \times B) \cap (A \times C)$. As $x \in A \times B$, we may write $x = (a, b)$, where $a \in A$ and $b \in B$. By definition, x is also a member of $A \times C$, and thus b is also a member of C. Since $b \in B$ and $b \in C$, we have $b \in B \cap C$, and thus $x \in A \times (B \cap C)$. Therefore $A \times (B \cap C) \supseteq (A \times B) \cap (A \times C)$.

As both containments are met, we therefore have equality: $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

2. Let $f(x) = x^2 - 3$ and let $g(x) = x^2 - 2$. Let $A = f(\mathbb{R})$ and let $B = g(\mathbb{R})$, the images of $\mathbb{R}$ under these functions. Prove that $B \subseteq A$ but $A \neq B$.

Solution: Suppose that $y \in B$. Then $y \in g(\mathbb{R})$, so there exists $x \in \mathbb{R}$ such that $g(x) = y$; that is, $x^2 - 2 = y$. Define $z = \sqrt{x^2 + 1}$. Note that z exists in $\mathbb{R}$, as $x^2 + 1$ is positive. Moreover, $f(z) = z^2 - 3 = x^2 + 1 - 3 = x^2 - 2 = y$. Hence, $y \in f(\mathbb{R})$, as we have found a real number z satisfying $f(z) = y$. Therefore, $y \in A$, and as y was chosen arbitrarily from B, we have $B \subseteq A$.
However, let us consider $y = -3$. Note that $y \in A$, as $f(0) = 0^2 - 3 = -3$. But $y \notin B$, as $g(x) = y$ would imply that $x^2 - 2 = -3$, and thus $x^2 = -1$. As $x^2 = -1$ has no real solution for x, we cannot have y as a member of B. Therefore, $A \not\subseteq B$. Therefore, $A \neq B$.

3. Let $A = \{x \in \mathbb{R} \mid -3 < x < 2\}$ and let $B = \{x \in \mathbb{R} \mid x^2 + x - 6 < 0\}$. Prove that $A = B$.

Important Note! You should not use calculus to prove this property. We *technically* don't know any calculus in this class.

Solution: Notice that $x^2 + x - 6 = (x+3)(x-2)$, and that this product can be negative if and only if exactly one of the factors $x + 3$ or $x - 2$ is negative. We use this fact to prove equality via double containment.
($\subseteq$) Suppose $x \in A$. Then $-3 < x < 2$, and therefore $0 < x+3 < 5$ and $-5 < x-2 < 0$. Hence, $x+3 > 0$ and $x-2 < 0$, and thus $(x+3)(x-2) < 0$. As noted above, this implies $x^2 + x - 6 < 0$, so $x \in B$, and thus $A \subseteq B$.

($\supseteq$) Suppose $x \in B$. Then as noted above, $(x+3)(x-2) < 0$, so exactly one of these factors is negative. We consider two cases:
Case 1: $x + 3 < 0$, $x - 2 > 0$. Then as $x + 3 < 0$, we have $x < -3$, and as $x - 2 > 0$, we have $x > 2$. There is no real number x that satisfies these two inequalities simultaneously, and hence this case is impossible.
Case 2: $x + 3 > 0$, $x - 2 < 0$. Then $x > -3$ and $x < 2$, and thus $-3 < x < 2$ and $x \in A$.

Therefore, if $x \in B$, we must have $x \in A$, so $B \subseteq A$.

As both containments hold, we have $A = B$.

4. Let $S \subseteq \mathbb{N}$ be a set that has the following property: $\forall n, m \in S$, $|n - m| \in S$. Let ℓ be the minimum of S (note: the minimum exists because of Well Ordering). Prove that every element of S is divisible by ℓ, using the Well-Ordering Principle.

   Solution: Suppose, to the contrary, that not every element of S is divisible by ℓ. Let $B = \{n \in S \mid n \text{ is not divisible by } \ell\}$. Then B is nonempty, and $\ell \notin B$. As ℓ is a lower bound for S, we must also have that ℓ is a lower bound for B. By the Well-Ordering Principle, B has a minimum; let $n = \min B$. Then as ℓ is a lower bound for B and n is the infimum of B, we must have $n > \ell$.

   By definition of S, then, $n - \ell \in S$. Moreover, $n - \ell < n$, and hence $n - \ell \notin B$. Thus, $n - \ell$ is divisible by ℓ, so there exists $k \in \mathbb{N}$ such that $n - \ell = \ell k$. But then $n = \ell(k+1)$, and hence n is divisible by ℓ.

   This is a contradiction, and hence it must be the case that n is divisible by ℓ for all $n \in S$.

5. Use the Well-Ordering Principle to formally prove something we have assumed all along: every integer $n \in \mathbb{N}$ is either even or odd.

   Solution: Suppose, for the sake of contradiction, that there is some $n \in \mathbb{N}$ that is neither even nor odd. Let N be the minimal such number.

   Note that $N \neq 1$, since $1 = 2 \cdot 0 + 1$ is odd.

   Now, $N - 1 \geq 1$ is a positive integer, and since it is not a counterexample it is either even or odd. We consider two cases.

   Case 1: $N - 1$ is even. Then $N - 1 = 2k$ for some $k \geq 0$, and hence $N = 2k + 1$ is odd, a contradiction.

   Case 2: $N - 1$ is odd. Then $N - 1 = 2k + 1$ for some $k \geq 0$, so $N = 2k + 2 = 2(k+1)$ is even, a contradiction.

   In either case, then, we achieve a contradiction, and hence we must have that there is no such counterexample. Therefore, every $n \in \mathbb{N}$ is either even or odd.

6. Let $f : X \to Y$ be a function. Suppose there exists $y_1, y_2 \in Y$, with $y_1 \neq y_2$, such that $f^{-1}(y_1) = f^{-1}(y_2)$. Prove that $y_1 \notin f(X)$ and $y_2 \notin f(X)$.

   Solution: Suppose that $f^{-1}(y_1) = f^{-1}(y_2)$, and also that $y_1 \in f(X)$. Then there exists $x \in X$ such that $f(x) = y_1$, and thus $x \in f^{-1}(y_1)$. But then we also have $x \in f^{-1}(y_2)$, and thus $f(x) = y_2$. Since $y_1 \neq y_2$, this is impossible. Therefore, if $f^{-1}(y_1) = f^{-1}(y_2)$, we must have $y_1 \notin f(X)$. By an identical argument, we also obtain $y_2 \notin f(X)$.

7. Let $\mathbb{Q}^+$ be the set of positive rational numbers. Let $f : \mathbb{Q}^+ \to \mathbb{N}$ be defined by $f(q) = n$, where n is the numerator of q. Is this a well-defined function? If so, prove it. If not, explain why not, and make an appropriate modification to produce a well-defined function.

Solution: No, this function is not well defined, as each rational number q can have more than one numerator. For example, if $q = 2 = 4/2$, we cannot know if the intended numerator is 2 or 4.

We can make this well-defined by choosing n to be the minimum positive numerator possible for q. Note that the Well-Ordering Principle implies such a minimum must exist, and it is now unique, so the function is well-defined under this definition.

8. Let $f : X \to Y$ be a function, and let $A \subseteq X$. Prove that for all $B \subseteq Y$, $f|_A^{-1}(B) = f^{-1}(B) \cap A$.

   Solution: We prove equality via double containment.

   ($\subseteq$) Suppose $x \in f|_A^{-1}(B)$. Then by definition, $x \in A$ and there exists $y \in B$ such that $f|_A(x) = y$. But $f|_A(x) = f(x)$, and hence since $f(x) \in B$, $x \in f^{-1}(B)$. Therefore, as $x \in A$ and $x \in f^{-1}(B)$, we have $x \in f^{-1}(B) \cap A$, so $f|_A^{-1}(B) \subseteq f^{-1}(B) \cap A$.

   ($\supseteq$) Let $x \in f^{-1}(B) \cap A$. Then $x \in f^{-1}(B)$, so $f(x) \in B$. Moreover, $x \in A$, so $f|_A$ is defined on x, and $f|_A(x) = f(x) \in B$. Therefore, $x \in f|_A^{-1}(B)$, and thus $f|_A^{-1}(B) \supseteq f^{-1}(B) \cap A$.

   Therefore, $f|_A^{-1}(B) = f^{-1}(B) \cap A$.

9. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function defined by $f(x, y) = x$, and let $g : \mathbb{R} \to \mathbb{R}^2$ be a function defined by $g(x) = (x, 0)$. Show that g is a right inverse for f, but not a left inverse. Show that g is not unique as a right inverse; that is, there exists another function h that is a right inverse to f and $h \neq g$.

   Solution: Notice that
   $$g \circ f(x, y) = g(x) = (x, 0),$$
   and
   $$f \circ g(x) = f(x, 0) = x.$$
   Thus, $f \circ g = id_{\mathbb{R}}$, so g is a right inverse for f. But $g \circ f \neq id_{\mathbb{R}^2}$, so g is not a left inverse for f.

   Define $h : \mathbb{R} \to \mathbb{R}^2$ by $h(x) = (x, 1)$. Note that h is also a right inverse for f, as $f \circ h(x) = f(x, 1) = x$, and $h \neq g$.

10. Prove that if f and g are both bijective, then $f \circ g$ is bijective, and $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

    Solution: Let $f : Y \to Z$ and $g : X \to Y$, both bijective, with inverses f^{-1} and g^{-1} respectively. We note that to show that $f \circ g$ is bijective, it is sufficient to show that $f \circ g$ has an inverse. Let $h = g^{-1} \circ f^{-1}$.

    Note that
    $$h \circ (f \circ g) = g^{-1} \circ f^{-1} \circ f \circ g = g^{-1} \circ id_Y \circ g = g^{-1} \circ g = id_X,$$
    and
    $$(f \circ g) \circ h = f \circ g \circ g^{-1} \circ f^{-1} = f \circ id_Y \circ f^{-1} = f \circ f^{-1} = id_Z.$$
    Therefore, as $h \circ (f \circ g)$ and $(f \circ g) \circ h$ are both identity functions, h is an inverse to $f \circ g$. We conclude that $f \circ g$ is bijective, and $(f \circ g)^{-1} = h = g^{-1} \circ f^{-1}$.

11. Determine the number of positive integers less than 2000 that are multiples of $4, 7$, and 13.

    Solution: For any d, let N_d denote the number of positive integers less than 2000 that are multiples of d. Note that $N_d = \lfloor \frac{1999}{d} \rfloor$. We define $N_{d_1,d_2,\ldots,d_k}$ to be the set of numbers that are divisible by all of $d_1, d_2, \ldots, d_k$. Note that to be divisible by 4 and 7 and 13, a number must be divisible by $4*7*13 = 364$. Hence, the number of positive integers less than 2000 that are multiples of 4, 7, and 13 is $N_{364} = \lfloor \frac{1999}{364} \rfloor = \lfloor 5.49 \rfloor = 4$.

12. Talking to one of your professors, you find out that he has been working at CMU for 24 years, and has taught 2 different courses each semester. You also find out that he has only taught 10 different courses. Your friend then immediately says that at least two of his semesters must have been the same. Is your friend right? How did he know?

    Solution: Yes, your friend is right. Each semester, the professor has taught 2 different courses. Since there are only 10 courses to choose from, there are $\binom{10}{2} = 45$ different ways he could teach a semester. But the professor has been teaching for 24 years, which amounts to 48 semesters. Thus, by the Pigeonhole Principle, at least two semesters must have been the same.

13. Let $n, m \in \mathbb{N}$. How many functions $f : [n] \to [m]$ are injections?

    Solution: Note that if $f : [n] \to [m]$ is an injection, we must have $m \geq n$, so if $m < n$, there are 0 injections from $[n] \to [m]$. If $m \geq n$, we consider the injection as follows. First, we select a value in $[m]$ to be $f(1)$. This can be done in m ways. Next, we select a value in $[m]$ to be $f(2)$. This number must be different from the value of $f(1)$, so there are $m - 1$ options for $f(2)$. Likewise, there are $m - 2$ options for $f(3)$, $m - 3$ options for $f(4)$, $\ldots$, $m-n+1$ options for $f(n)$. Hence, if $m \geq n$, there are $m(m-1)(m-2)\ldots(m-n+1) = \frac{m!}{(m-n)!}$ injections $f : [n] \to [m]$.

14. Let $n \in \mathbb{N}$. How many surjective functions are there from $[n]$ to $[3]$? (Hint: first think about counting functions that are NOT surjective)

    Solution: Recall that the total number of functions from $[n]$ to $[3]$ is 3^n.

    Let's count the number of those functions that are NOT surjective. If a function is not surjective, there must be some element of the codomain that is not in the range.

    Define

    $$\begin{aligned} X_1 &= \{f : [n] \to [3] \mid 1 \notin f([n])\} \\ X_2 &= \{f : [n] \to [3] \mid 2 \notin f([n])\} \\ X_3 &= \{f : [n] \to [3] \mid 3 \notin f([n])\} \end{aligned}$$

    Then the number of functions from $[n]$ to $[3]$ that are not surjective is $X_1 \cup X_2 \cup X_3$. We can count this by Inclusion-Exclusion.

We have

$$\begin{aligned}|X_1 \cup X_2 \cup X_3| &= |X_1| + |X_2| + |X_3| - |X_1 \cap X_2| - |X_1 \cap X_3| \\ &\quad -|X_2 \cap X_3| + |X_1 \cap X_2 \cap X_3| \\ &= 2^n + 2^n + 2^n - 1^n - 1^n - 1^n + 0^n \\ &= 3 \cdot 2^n - 3.\end{aligned}$$

Note that here, $|X_1| = 2^n$ since $X_1 = \{f : [n] \to \{2, 3\}\}$, and likewise for the other counts.

Therefore, the number of surjective functions from $[n]$ to $[3]$ is $3^n - (3 \cdot 2^n - 3)$.

15. You are planning to volunteer at a homeless shelter for 3 days in May (hey, good for you!). You can pick any 3 days, but you don't ever want to work two days in a row. How many different ways can you choose the days?

    Solution: Let us view the days in May as follows:

    $$\underline{\quad a_1 \quad} \cdot \underline{\quad a_2 \quad} \cdot \underline{\quad a_3 \quad} \cdot \underline{\quad a_4 \quad}$$

    Here, the dots represent workdays, and the numbers a_1, a_2, a_3, a_4 represent the number of days in between. So, for example, if you worked on May 3, 6, and 31, we would have $a_1 = 2$, $a_2 = 2$, $a_3 = 24$, and $a_4 = 0$. The rules of the problem state that $a_2 \geq 1$ and $a_3 \geq 1$. Moreover, we must have $a_1 + a_2 + a_3 + a_4 = 28$, since they represent all of the days you do not work in May.

    This is similar to the partition problem in Homework 9, except that in that assignment, we wanted all the parts to be ≥ 0, not ≥ 1 sometimes. So we can shift, and rewrite this as

    $$\underline{\quad a_1 \quad} \cdot \underline{\quad b_2 + 1 \quad} \cdot \underline{\quad b_3 + 1 \quad} \cdot \underline{\quad a_4 \quad},$$

    where now $a_1, b_2, b_3, a_4 \geq 0$, and $a_1 + b_2 + b_3 + a_4 = 26$.

    From homework, we have that the number of partitions of 26 into 4 parts is precisely $\binom{26+4-1}{4-1} = \binom{29}{3}$.

    Hence, the number of ways to create the schedule is $\binom{29}{3}$

16. How many flags can you make that consist of three horizontal stripes of the colors red, white, blue, green, and black, where consecutive stripes must be different colors?

    Solution: There are 5 total colors.

    We can choose the first stripe in one of 5 ways.

    The second stripe can be any color except the color we chose already, so it can be any of 4 colors.

    The final stripe can be any color except the same as the middle, so it can be any of 4 colors.

    Hence there are $5 \cdot 4 \cdot 4 = 80$ possible flags.

17. Prove, by counting in two ways, that for any $n, s \in \mathbb{N}$, we have

    $$\sum_{k=1}^{n} \binom{n}{n-k}\binom{s-1}{k-1} = \binom{n+s-1}{n-1}.$$

Solution: Let X be a set of $n+s-1$ people, of which n are women and $s-1$ are men. We wish to select a group of $n-1$ people from X to be on a committee. We note that this can be done in $\binom{n+s-1}{n-1}$ ways.
We can count the number of ways to form the committee in another way. We first choose some number of men to be on the committee. The number of men we choose can be anything from 0 to $n-1$. Let $k-1$ be the number of men chosen, so k ranges from 1 to n. Then we need to choose $n-1-(k-1)=n-k$ women in order for the committee to have a total of $n-1$ people. Hence, for each choice of k, we can form a committee having $k-1$ men and $n-k$ women in $\binom{n}{n-k}\binom{s-1}{k-1}$ different ways. Adding up this count for every possible value of k yields that the number of ways to form a committee of $n-1$ people is $\sum_{k=1}^{n}\binom{n}{n-k}\binom{s-1}{k-1}$. Therefore, since both sides of the equality count the same set, they are equal.

18. Prove using counting in two ways that for all $k, n \in \mathbb{N}$,

$$\sum_{j=k}^{n}\binom{j}{k}=\binom{n+1}{k+1}.$$

(Hint: think about a subset in terms of its largest element)

Solution: The number of subsets of $[n+1]$ of size $k+1$ is exactly $\binom{n+1}{k+1}$.
On the other hand, let us partition the subsets of $[n+1]$ of size $k+1$ into sets $A_{k+1}, A_{k+2}, \ldots, A_{n+1}$, where

$$A_j=\left\{S \in \binom{[n+1]}{k+1} \mid \max(S)=j\right\}.$$

Clearly $\cup A_j=\binom{[n+1]}{k+1}$ and the A_j are pairwise disjoint, so the A_j form a pairwise disjoint finite partition of $\binom{[n+1]}{k+1}$. Thus, $\left|\binom{[n+1]}{k+1}\right|=\sum\limits_{j=k+1}^{n+1}|A_j|$.
Now, given $S \in A_j$, we know that the largest element of S is j. Hence, there are k elements of S that can be chosen in any way from $[j-1]$. Therefore, $|A_j|=\binom{j-1}{k}$.
Hence, we have

$$\begin{aligned}\sum_{j=k+1}^{n+1}|A_j| &= \sum_{j=k+1}^{n+1}\binom{j-1}{k} \\ &= \sum_{j=k}^{n}\binom{j}{k},\end{aligned}$$

where the last line follows from reindexing.
Therefore,

$$\sum_{j=k}^{n}\binom{j}{k}=\binom{n+1}{k+1}.$$

19. Prove that

$$\sum_{k=0}^{n}\sum_{j=0}^{n-k}\frac{n!}{k!j!(n-k-j)!}=3^n.$$

Solution: Recall the binomial theorem: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.
We thus have

$$\begin{aligned}
\sum_{k=0}^{n} \sum_{j=0}^{n-k} \frac{n!}{k!j!(n-k-j)!} &= \sum_{k=0}^{n} \sum_{j=0}^{n-k} \frac{n!(n-k)!}{(n-k)!k!j!((n-k)-j)!} \\
&= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} 1^k \sum_{j=0}^{n-k} \frac{(n-k)!}{j!((n-k)-j)!} 1^j 1^{(n-k)-j} \\
&= \sum_{k=0}^{n} \binom{n}{k} 1^k \sum_{j=0}^{n-k} \binom{n-k}{j} 1^j 1^{(n-k)-j} \\
&= \sum_{k=0}^{n} \binom{n}{k} 1^k (2)^{n-k} \\
&= 3^n.
\end{aligned}$$

20. Let X be an infinite set, and let $S \subset X$ be finite. Prove that $|X| = |X \backslash S|$.

Solution: We work by induction on $|S|$.
First, suppose that $|S| = 1$, so that $S = \{x_0\}$. Since X is infinite, we must have that $|X| \geq |\mathbb{N}|$, and thus there exists an injection $f : \mathbb{N} \to X$. We consider two cases, according as whether $x_0 \in f(\mathbb{N})$ or $x_0 \notin f(\mathbb{N})$.
Case 1: $x_0 \in f(\mathbb{N})$. Let $n \in \mathbb{N}$ such that $x_0 = f(n)$. Define a function $g : X \to X \backslash \{x_0\}$ as follows:

$$g(x) = \begin{cases} x & x \notin f(\mathbb{N}) \\ f(k) & x \in f(\mathbb{N}) \text{ and } f^{-1}(x) = k < n \\ f(k+1) & x \in f(\mathbb{N}) \text{ and } f^{-1}(x) = k \geq n \end{cases}$$

That is to say, the function g acts like the identity function in most cases, except for shifting down along the function f to omit the element x_0. This is clearly a bijection, so $|X| = |X \backslash \{x_0\}|$.
Case 2: $x_0 \notin f(\mathbb{N})$. Define a new function $h : \mathbb{N} \to X$ by $h(n) = x_0$ if $n = 1$, and $h(n) = f(n-1)$ if $n > 1$. Then we can apply Case 1.
Hence, if $|S| = 1$, it is true that $|X| = |X \backslash S|$.
Now, suppose the result is true for a set of size n. Consider a set S of size $n+1$, and write $S = \{s_1, s_2, \ldots, s_n, s_{n+1}\}$. Then by the induction hypothesis, we have that $|X| = |X \backslash \{s_1, s_2, \ldots, s_n\}|$, and by the base case, we have that $|(X \backslash \{s_1, s_2, \ldots, s_n\}| = |X \backslash \{s_1, s_2, \ldots, s_n\} \backslash \{s_{n+1}\}|$. Hence,

$$|X| = |X \backslash \{s_1, s_2, \ldots, s_n\} \backslash \{s_{n+1}\}| = |X \backslash S|.$$

21. Let A be finite, B be countably infinite, C be countable, and D be uncountable. What can you say about the cardinalities of the following sets?

(a) $A \cup B$

(b) $A \cup C$

(c) $A \cap C$

(d) $B \cap D$

(e) $D \backslash B$

(f) $B \backslash D$

(g) $C \cup D$

Solution:

(a) $A \cup B$ countably infinite
(b) $A \cup C$ countable
(c) $A \cap C$ finite
(d) $B \cap D$ countable (but we cannot know if it is infinite or finite)
(e) $D \backslash B$ uncountable
(f) $B \backslash D$ countable (but we cannot know if it is infinite or finite)
(g) $C \cup D$ uncountable

22. Explain Cantor's diagonalization argument.

Solution: Solutions may vary

23. Let $B \subseteq \mathbb{R}$ be a set of numbers having the properties

- $b \geq 0 \ \forall b \in B$
- If $S \subseteq B$ is a finite set, then $\sum_{b \in S} b < 3$.

Prove that B is countable. (Hint: how many elements of B can be greater than $1/n$ for each $n \in \mathbb{N}$?)

Solution: For each $n \in \mathbb{N}$, we must have that the number of elements of B greater than $1/n$ is at most $3n$, since every finite subset of B sums to no more than 3.

Let $A_n = \{x \in B \mid x \geq \frac{1}{n}\}$. The above argument shows that A_n is finite for all n.

Now, let $x \in B$. Then either $x = 0$ or $x > 0$. If $x > 0$, then by the Archimedean principle there exists some n such that $x > \frac{1}{n}$. Therefore, $x \in A_n$ for some n.

Hence $B \subseteq \{0\} \cup \cup_{n=1}^{\infty} A_n$, a countable union of countable sets. Therefore, B is countable.