21-127 Exam 2 Review Materials

Mary Radcliffe

Major topics for your second exam include:

- Basics of set theory. Including: understanding of unions, intersections, power sets, Cartesian Products, and how to prove two sets are equal.
- DeMorgan's Laws for sets
- Well-Ordering Principle
- Basics of functions. Including: definition of function as well as key terms (domain, codomain, graph, well-defined, composition, image, preimage). Understanding of how to prove that a function is well-defined, and manipulate other basic properties.
- More sophisticated function stuff: understanding of injectivity, surjectivity, bijectivity. Understanding of how to prove a function satisfies any of these criteria. Knowing how these properties relate to left-, right-, and two-side inverses.
- Finite counting: Definitions and theorems. Inclusion/Exclusion. Basic theorems about counting unions, intersections, products, etc.
- Basic combinatorial tools: permutations, $\binom{X}{k}$, binomial coefficients. Counting techniques. Counting in 2 ways, bijections, using Cartesian products.
- Binomial Theorem, applications.
- Infinite sets. Definitions of countable/uncountable. Theorems about how to determine if a set is countably infinite. Know which sets are which sizes, and how to tell.

Some practice problems:

1. Let A, B, C be subsets of the universe Z. Prove each of the following identities:
   (a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
   (b) $(A \cap B) \cup (A \cap B^c) = A$
   (c) $(A^c \cap Z)^c = A$
   (d) $A \times (B \cap C) = (A \times B) \cap (A \times C)$
2. Let $f(x) = x^2 - 3$ and let $g(x) = x^2 - 2$. Let $A = f(\mathbb{R})$ and let $B = g(\mathbb{R})$, the images of $\mathbb{R}$ under these functions. Prove that $B \subseteq A$ but $A \neq B$.
3. Let $A = \{x \in \mathbb{R} \mid -3 < x < 2\}$ and let $B = \{x \in \mathbb{R} \mid x^2 + x - 6 < 0\}$. Prove that $A = B$.

4. Let $S \subseteq \mathbb{N}$ be a set that has the following property: $\forall n, m \in S, |n-m| \in S$. Let ℓ be the minimum of S (note: the minimum exists because of Well Ordering). Prove that every element of S is divisible by ℓ, using the Well-Ordering Principle.

5. Use the Well-Ordering Principle to formally prove something we have assumed all along: every integer $n \in \mathbb{N}$ is either even or odd.

6. Let $f : X \to Y$ be a function. Suppose there exists $y_1, y_2 \in Y$, with $y_1 \neq y_2$, such that $f^{-1}(y_1) = f^{-1}(y_2)$. Prove that $y_1 \notin f(X)$ and $y_2 \notin f(X)$.

7. Let $\mathbb{Q}^+$ be the set of positive rational numbers. Let $f : \mathbb{Q}^+ \to \mathbb{N}$ be defined by $f(q) = n$, where n is the numerator of q. Is this a well-defined function? If so, prove it. If not, explain why not, and make an appropriate modification to produce a well-defined function.

8. Let $f : X \to Y$ be a function, and let $A \subseteq X$. Prove that for all $B \subseteq Y$, $f|_A^{-1}(B) = f^{-1}(B) \cap A$.

9. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function defined by $f(x, y) = x$, and let $g : \mathbb{R} \to \mathbb{R}^2$ be a function defined by $g(x) = (x, 0)$. Show that g is a right inverse for f, but not a left inverse. Show that g is not unique as a right inverse; that is, there exists another function h that is a right inverse to f and $h \neq g$.

10. Determine the number of positive integers less than 2000 that are multiples of $4, 7$, and 13.

11. Talking to one of your professors, you find out that he has been working at CMU for 30 years, and has taught 2 different courses each semester. You also find out that he has only taught 10 different courses. Your friend then immediately says that at least two of his semesters must have been the same. Is your friend right? How did he know?

12. Let $n, m \in \mathbb{N}$. How many functions $f : [n] \to [m]$ are injections?

13. Let $n \in \mathbb{N}$. How many surjective functions are there from $[n]$ to $[3]$? (Hint: first think about counting functions that are NOT surjective)

14. You are planning to volunteer at a homeless shelter for 3 days in May (hey, good for you!). You can pick any 3 days, but you don't ever want to work two days in a row. How many different ways can you choose the days?

15. How many flags can you make that consist of three horizontal stripes of the colors red, white, blue, green, and black, where consecutive stripes must be different colors?

16. Prove, by counting in two ways, that for any $n, s \in \mathbb{N}$, we have
$$\sum_{k=1}^{n} \binom{n}{n-k} \binom{s-1}{k-1} = \binom{n+s-1}{n-1}.$$

17. Prove using counting in two ways that for all $k, n \in \mathbb{N}$,
$$\sum_{j=k}^{n} \binom{j}{k} = \binom{n+1}{k+1}.$$
(Hint: think about a subset in terms of its largest element)

18. Prove that

$$\sum_{k=0}^{n} \sum_{j=0}^{n-k} \frac{n!}{k!j!(n-k-j)!} = 3^n.$$

19. Let X be an infinite set, and let $S \subset X$ be finite. Prove that $|X| = |X \backslash S|$.

20. Let A be finite, B be countably infinite, C be countable, and D be uncountable. What can you say about the cardinalities of the following sets?

    (a) $A \cup B$

    (b) $A \cup C$

    (c) $A \cap C$

    (d) $B \cap D$

    (e) $D \backslash B$

    (f) $B \backslash D$

    (g) $C \cup D$

21. Explain Cantor's diagonalization argument.

22. Let $B \subseteq \mathbb{R}$ be a set of numbers having the properties

    - $b \geq 0 \ \forall b \in B$
    - If $S \subseteq B$ is a finite set, then $\sum_{b \in S} b < 3$.

    Prove that B is countable. (Hint: how many elements of B can be greater than $1/n$ for each $n \in \mathbb{N}$?)