21-127 Exam 1 Review Problems: Solutions

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1. Prove that $(\neg(p \wedge q)) \wedge r$ is logically equivalent to $\neg((p \vee (\neg r)) \wedge (q \vee (\neg r)))$.

   Solution: We consider the following two truth tables:

p	q	r	$p \wedge q$	$\neg(p \wedge q)$	$(\neg(p \wedge q)) \wedge r$
T	T	T	T	F	F
F	T	T	F	T	T
T	F	T	F	T	T
F	F	T	F	T	T
T	T	F	T	F	F
F	T	F	F	T	F
T	F	F	F	T	F
F	F	F	F	T	F

p	q	r	$\neg r$	$p \vee (\neg r)$	$q \vee (\neg r)$	$(p \vee (\neg r)) \wedge (q \vee (\neg r))$	$\neg((p \vee (\neg r)) \wedge (q \vee (\neg r)))$
T	T	T	F	T	T	T	F
F	T	T	F	F	T	F	T
T	F	T	F	T	F	F	T
F	F	T	F	F	F	F	T
T	T	F	T	T	T	T	F
F	T	F	T	T	T	T	F
T	F	F	T	T	T	T	F
F	F	F	T	T	T	T	F

   Therefore, as the columns corresponding to $(\neg(p \wedge q)) \wedge r$ and $\neg((p \vee (\neg r)) \wedge (q \vee (\neg r)))$ are equal, the two formulae are logically equivalent.

2. Prove that $(p \implies q) \wedge (\neg p \implies q)$ is logically equivalent to q.

   Solution: Again, we consider the following truth table:

p	q	$\neg p$	$p \Rightarrow q$	$(\neg p) \Rightarrow q$	$(p \Rightarrow q) \wedge ((\neg p) \Rightarrow q)$
T	T	F	T	T	T
F	T	T	T	T	T
T	F	F	F	T	F
F	F	T	T	F	F

   Note that the column for q and for $(p \Rightarrow q) \wedge ((\neg p) \Rightarrow q)$ are identical, and hence the two propositions are logically equivalent.

3. Suppose that x and y both have the range of real numbers. Explain the difference between the following two statements.

   - $\forall x, \exists y, x^2 = y$

- $\exists y, \forall x, x^2 = y$

  Solution: The first statement says that for any $x \in \mathbb{R}$, a real number y can be chosen (possibly depending upon x) in order to satisfy the equation $x^2 = y$.

  The second statement says that there is a real number y, that can be chosen independently of x, that satisfies the equation $x^2 = y$ for every choice of $x \in \mathbb{R}$.

  Clearly the first statement is true, and the second is false.

4. Write the following statement as a propositional formula. Then prove it.

   Let $n \in \mathbb{N}$. If n^2 is divisible by 4 and n^3 is divisible by 27, then n is divisible by 6.

   Solution: Let $p(n)$ be the proposition "n^2 is divisible by 4;" let $q(n)$ be the proposition "n^3 is divisible by 27;" let $r(n)$ be the proposition "n is divisible by 2;" and let $t(n)$ be the proposition "n is divisible by 3." The statement can then be written as

   $$\forall n, p(n) \wedge q(n) \Rightarrow r(n) \wedge t(n).$$

   Now, let us prove the statement. Let $n \in \mathbb{N}$ be such that $p(n) \wedge q(n)$ is true.

   Write $n = 2k + r$, where $r = 0$ or 1, according to the division theorem. Then $n^2 = (2k+r)^2 = 4k^2+4kr+r^2 = 4(k^2+kr)+r^2$. Since $p(n)$ is true, it must be the case that n^2 is divisible by 4, so r^2 must be divisible by 4. But r^2 is either 0 or 1, so it must be that $r^2 = 0$ and $r = 0$. Therefore, $n = 2k$, and hence $r(n)$ is true. Hence, we conclude $p(n) \wedge q(n) \Rightarrow r(n)$.

   Similarly, write $n = 3\ell + r$, where $r = 0, 1$, or 2, according to the division theorem. Then $n^3 = (3\ell + r)^3 = 27\ell^3 + 27\ell^2 r + 9\ell r^2 + r^3 = 27(\ell^3 + \ell^2 r) + r^2(9\ell + r)$. As $q(n)$ is true, we must have that 27 divides n^3, and thus 27 must divide $r^2(9\ell+r)$. We consider three cases, according to the value of r.

   Case 1: $r = 0$. Then $r^2(9\ell+r) = 0$ and is divisible by 27. **Case 2:** $r = 1$. Then $r^2(9\ell + r) = 9\ell + 1$, which is not divisible by 27 for any choice of ℓ. **Case 3:** $r = 2$. Then $r^2(9\ell+r) = 4(9\ell+2)$. Note that as 27 and 4 share no factors, this can only be divisible by 27 if $9\ell + 2$ is divisible by 27. But this is impossible, and hence this is not divisible by 27 for any choice of ℓ.

   Therefore, we must have $r = 0$, and thus $n = 3\ell$. Hence $t(n)$ is true, and we conclude that $p(n) \wedge q(n) \Rightarrow t(n)$.

   As both $p(n) \wedge q(n) \Rightarrow r(n)$ and $p(n) \wedge q(n) \Rightarrow t(n)$ are true, we therefore have $p(n) \wedge q(n) \Rightarrow r(n) \wedge t(n)$ is also true.

5. Write the following statement as a propositional formula. Then prove it.

   There is no integer value of x satisfying $0x = 1$.

   Solution: Let x be from the range of integers, take $p(x)$ to be the proposition "$0x = 1$." The statment then can be written as

   $$\neg(\exists x, p(x)).$$

   To prove this statement, we note that it is equivalent by De Morgan's laws to prove the proposition $\forall x, \neg p(x)$. We consider two cases. First, if $x = 0$, then $0x = 0 \neq 1$. Second, if $x \neq 0$, then we note that $1/x \neq 0$. Therefore, if $0x = 1$, it would be the case that $1/x = 0$, which is impossible.

Therefore, no value of x can make $p(x)$ true. Thus, $\forall x, \neg p(x)$.

6. Suppose $p(x)$ is a polynomial that can be written as

$$p(x) = (x - a_1)(x - a_2)\dots(x - a_n).$$

Prove that α is a root of $p(x)$ if and only if $\alpha = a_i$ for some i with $1 \leq i \leq n$.

Solution: We first observe the following:

Lemma 1. *If $n \geq 1$ and $a_1, a_2, \dots, a_n \in \mathbb{R}$ satisfy $a_1 a_2 \dots a_n = 0$, then $a_i = 0$ for at least one value of i between 1 and n.*

Proof of Lemma. We proceed by induction on n. We consider the base cases as $n = 1$ and $n = 2$.

For $n = 1$, notice that $a_1 = 0$, and we are done.

For $n = 2$, suppose that $a_1 a_2 = 0$. If $a_1 = 0$, we are done. If not, we may divide both sides of the expression by a_1, and therefore obtain $a_2 = 0/a_1 = 0$.

Now, suppose that for some $n \in \mathbb{N}$, the result is known. Let us consider the $(n+1)$-fold product $a_1 a_2 \dots a_{n+1} = 0$. Take $a_2' = a_2 \dots a_{n+1}$. Then $a_1 a_2 \dots a_{n+1} = a_1 a_2' = 0$, and hence by the case that $n = 2$ above, we must have either $a_1 = 0$ or $a_2' = 0$. If $a_1 = 0$, then we are done. If not, then $a_2' = 0$, and by the inductive hypothesis, it must be the case that at least one of $a_2, \dots, a_{n+1}$ is 0.

Therefore, for any $n \in \mathbb{N}$, if $a_1 a_2 \dots a_n = 0$, then $a_i = 0$ for at least one value of i between 1 and n. □

Now, let us return to the polynomials. Let $p(x) = (x - a_1)(x - a_2)\dots(x - a_n)$. We consider first the forward direction.

If α is a root of $p(x)$, then $p(\alpha) = (\alpha - a_1)(\alpha - a_2)\dots(\alpha - a_n) = 0$. By the Lemma, then, we must have that $\alpha - a_i = 0$ for some choice of i, that is, $\alpha = a_i$ for some choice of i.

For the converse, suppose that $\alpha = a_i$ for some choice of i. Then clearly $p(\alpha) = 0$, as $p(\alpha) = (\alpha - a_1)(\alpha - a_2)\dots(\alpha - a_n)$.

Hence, α is a root of $p(x)$ if and only if $\alpha = a_i$ for some i.

7. Let $a, u, b, v, d \in \mathbb{N}$. For each of the following, determine if it is true or false. Prove that your answer is correct.

(a) If $d|a$ and $d|b$, then $d|(au + bv)$.

Solution: True. Let $n, m \in \mathbb{Z}$ be such that $a = dn$ and $b = dm$. Then $au + bv = dnu + dmv = d(nu + mv)$, and therefore $d|(au + bv)$.

(b) If $d|(au + bv)$ then $d|a$ and $d|b$.

Solution: False. Consider, for example, $a = 1, u = 3, b = 1, v = 3, d = 2$. Notice that d does not divide either a or b, but d does divide $au + bv = 6$.

(c) If $d|a$ and d does not divide b or v, then d does not divide $au + bv$.

Solution: False. Consider $d = 6, a = 6, b = 2, v = 3$, and $u = 1$. Then $d|a$, d does not divide either b or v, but $au+bv = 12$, which is divisible by $6 = d$.

(d) If d does not divide any of a, u, b, v, then d does not divide $au + bv$.

Solution: False. Take $a = 2, u = 3, b = 2, v = 3, d = 6$, and note that d does not divide any of a, u, b, or v, but it does divide $au + bv = 12$.

8. Find all real solutions to the equation

$$\sqrt{x+10} + \sqrt{x+5} = 5.$$

Prove that your answer is correct.

Solution: First consider:

$$\begin{aligned}
x \text{ is a solution to the equation} \quad &\Rightarrow \quad \sqrt{x+10} + \sqrt{x+5} = 5 \\
&\Rightarrow \quad (\sqrt{x+10} + \sqrt{x+5})^2 = 5^2 \\
&\Rightarrow \quad x + 10 + 2\sqrt{(x+10)(x+5)} + x + 5 = 25 \\
&\Rightarrow \quad \sqrt{x^2 + 15x + 50} = 5 - x \\
&\Rightarrow \quad x^2 + 15x + 50 = x^2 - 10x + 25 \\
&\Rightarrow \quad 25x = -25 \\
&\Rightarrow \quad x = -1.
\end{aligned}$$

Moreover,

$$\begin{aligned}
x = -1 \quad &\Rightarrow \quad \sqrt{x+10} + \sqrt{x+5} = \sqrt{9} + \sqrt{4} = 5 \\
&\Rightarrow \quad x \text{ is a solution to the equation .}
\end{aligned}$$

Therefore, x is a solution to the given equation if and only if $x = -1$.

9. Use the method of proof by contradiction to prove the AM-GM inequality: if $a, b \in \mathbb{R}$ with $a, b > 0$, then $\frac{a+b}{2} \geq \sqrt{ab}$.

Solution: Suppose for the sake of contradiction that $\frac{a+b}{2} < \sqrt{ab}$ for some choice of $a, b \in \mathbb{R}$ with $a, b > 0$. Let us square both sides, to obtain

$$\frac{(a+b)^2}{4} < ab \quad \Rightarrow \quad a^2+2ab+b^2 < 4ab \quad \Rightarrow \quad a^2-2ab+b^2 < 0.$$

But $a^2 - 2ab + b^2 = (a-b)^2$, and hence we have that $(a-b)^2 < 0$. This is impossible, as the square of any real number is nonnegative. Therefore, we cannot have $a, b \in \mathbb{R}$ with $a, b > 0$ and having $\frac{a+b}{2} < \sqrt{ab}$.
Thus, for all $a, b \in \mathbb{R}$ with $a, b > 0$, we have $\frac{a+b}{2} \geq \sqrt{ab}$.

10. Suppose that $a, b \in \mathbb{Z}$. Prove by contradiction that if $4|(a^2 - 3b^2)$ then at least one of a, b is even.

Solution: Suppose for the sake of contradiction that $4|(a^2 - 3b^2)$ but that both a, b are odd. Then there exist integers k, j, ℓ such that $4k = a^2 - 3b^2$, and $a = 2j + 1$ and $b = 2\ell + 1$. Then we have

$$\begin{aligned} 4k &= (2j+1)^2 - 3(2\ell+1)^2 \\ &= 4j^2 + 4j + 1 - 12\ell^2 - 12\ell - 3 \\ &= 4(j^2 + j - 3\ell^2 - 3\ell) - 2. \end{aligned}$$

By rearranging, we thus obtain $2 = 4(j^2 + j - 3\ell^2 - 3\ell - k)$, and hence 2 is divisible by 4. This is clearly impossible, and therefore the original assumption is impossible.

Thus, if $4|(a^2 - 3b^2)$, we must have that at least one of a, b is even.

11. Prove, by contradiction, the following: $\forall x \in \mathbb{R}$, if $x \geq 1$, then $\sqrt{x} \leq x$.

    Solution: Suppose, for the sake of contradiction, that $x \in \mathbb{R}$ has $x \geq 1$ but $\sqrt{x} > x$. We then obtain

$$\begin{aligned} \sqrt{x} > x &\Rightarrow 1 > \frac{x}{\sqrt{x}} \\ &\Rightarrow 1 > \sqrt{x} \\ &\Rightarrow 1 \cdot 1 > \sqrt{x} \cdot \sqrt{x} = x \\ &\Rightarrow x < 1. \end{aligned}$$

    This is in direct contradiction to the assumption that $x \geq 1$.

    Therefore, if $x \in \mathbb{R},\ x \geq 1$, we must have that $\sqrt{x} < x$.

12. On a certain island, each inhabitant always lies or always tells the truth. Calvin and Beatrice live on the island.

    Calvin says: "Exactly one of us is lying."

    Beatrice says: "Calvin is telling the truth."

    Determine who is telling the truth and who is lying. Prove that your answer is correct.

    Solution: Both Calvin and Beatrice are liars.

    To prove this, let us suppose that Beatrice is telling the truth. Then her statement "Calvin is telling the truth" is true, and hence Calvin is also telling the truth. Hence, Calvin's statement that "Exactly one of us is lying" is also true. But we already have established that neither Calvin nor Beatrice is lying, and hence this is impossible.

    Therefore, it must be the case that Beatrice is lying. Hence, her statement "Calvin is telling the truth" is false, and thus Calvin must also be lying.

    Hence, Calvin and Beatrice are both liars.

13. Let $n \in \mathbb{N}$ with $n \geq 2$. Suppose that for all $k \in \mathbb{N}$ with $2 \leq k \leq \sqrt{n}$, k does not divide n. Prove that n is prime.

    Solution: Suppose, for contrapositive, that $n \in \mathbb{N}$ with $n \geq 2$ and n is not prime. Then there exist $2 \leq a, b < n$ having $ab = n$. If $a \leq \sqrt{n}$, then we are done, hence let us suppose that $a > \sqrt{n}$. Then we have $b < \frac{n}{\sqrt{n}}$, and thus in either case one of

a, b is an integer between 2 and $\sqrt{n}$ that divides n. Therefore, by contrapositive, if no integers k with $2 \leq k \leq \sqrt{n}$ divide n, we must have that n is prime.

14. Prove that for any $n \in \mathbb{N}$, $\displaystyle\sum_{k=0}^{n} k^3 = \left(\sum_{k=0}^{n} k\right)^2$.

    Solution: Recall from lecture that $\displaystyle\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$. Thus, we need prove that $\displaystyle\sum_{k=0}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2 = \frac{n^2(n+1)^2}{4}$.

    We work by induction on n. Note that for $n = 0$, the result is immediate, as both the left and right hand side of the equation are 0.

    Suppose that for some n, it is known that $\displaystyle\sum_{k=0}^{n} k^3 = \frac{n^2(n+1)^2}{4}$.

    Consider, then,

$$
\begin{aligned}
\sum_{k=0}^{n+1} k^3 &= (n+1)^3 + \sum_{k=0}^{n} k^3 \\
&= (n+1)^3 + k^3 = \frac{n^2(n+1)^2}{4} \\
&= (n+1)^2\left(n+1+\frac{n^2}{4}\right) \\
&= (n+1)^2\left(\frac{n^2+4n+4}{4}\right) \\
&= (n+1)^2\frac{(n+2)^2}{4},
\end{aligned}
$$

    as desired.

    Therefore, by induction, $\displaystyle\sum_{k=0}^{n} k^3 = \frac{n^2(n+1)^2}{4}$ for all $n \in \mathbb{N}$.

15. Suppose $a_{n+1} = 5a_n - 6a_{n-1}$ for any $n \geq 1$, with $a_0 = 1$ and $a_1 = 1$. Prove that $a_n = 2^{n+1} - 3^n$ for all $n \in \mathbb{N}$.

    Solution: We work by strong induction on n. Notice, for the cases of $n = 0$ and $n = 1$, the result is immediate.

    Suppose that $a_k = 2^{k+1} - 3^k$ for all k satisfying $1 \leq k \leq n$.

    Then

$$
\begin{aligned}
a_{n+1} &= 5a_n - 6a_{n-1} \\
&= 5(2^{n+1} - 3^n) - 6(2^n - 3^{n-1}) \\
&= 10 * 2^n - 6 * 2^n - 15 * 3^{n-1} + 6 * 3^{n-1} \\
&= 4 * 2^n - 9 * 3^{n-1} \\
&= 2^{n+1} - 3^{n+1}.
\end{aligned}
$$

    Therefore, by strong induction, for all $n \in \mathbb{N}$, we have $a_n = 2^{n+1} - 3^n$.

16. Let $n \in \mathbb{N}$, with $n \geq 1$. Prove that $11^n - 6$ is divisible by 5.

Solution: We work by induction on n. Notice the base case, where $n = 1$, is immediate, since $11^1 - 6 = 5$.

Suppose now that the result is known for some n. Write $11^n - 6 = 5k$ for some $k \in \mathbb{Z}$. Consider $11^{n+1} - 6 = 11 * 11^n - 6 = 11 * (11^n - 6) + 66 - 6 = 11 * 5k + 60 = 5(11k + 12)$. Therefore, $11^{n+1} - 6$ is divisible by 5.

By induction, then $11^n - 6$ is divisible by 5 for all $n \geq 1$.

17. Let $n \in \mathbb{N}$. Prove that $\displaystyle\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$.

Solution: We work by induction on n. Notice that for $n = 0$, the left hand side of the equation is the empty sum, which is 0, as is the right hand side.

Now, suppose that for some n it is known that $\displaystyle\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$. Consider

$$\begin{aligned}
\sum_{k=1}^{n+1} \frac{1}{k(k+1)} &= \frac{1}{(n+1)(n+2)} + \sum_{k=1}^{n} \frac{1}{k(k+1)} \\
&= \frac{1}{(n+1)(n+2)} + \frac{n}{n+1} \\
&= \frac{1 + n(n+2)}{(n+1)(n+2)} \\
&= \frac{(n+1)^2}{(n+1)(n+2)} = \frac{n+1}{n+2}.
\end{aligned}$$

Therefore, for any $n \in \mathbb{N}$, we have $\displaystyle\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$.

18. Let $n, m \in \mathbb{N}$, with $n, m \geq 1$. A chocolate bar is made up of an $n \times m$ grid of squares. You break the chocolate bar into 1×1 pieces by iteratively breaking along a grid line. How many times must you make a break? Prove that your answer is correct.

Solution: We claim that you must make $nm - 1$ total breaks.

Our proof proceeds by strong induction on the product nm. Note that the smallest possible product is $nm = 1$, in which case $n = 1$ and $m = 1$. This serves as our base case. In this case, the chocolate bar has only one square, and hence 0 breaks are needed.

Now, suppose that we have an $n \times m$ chocolate bar, where it is known that for any $k \times j$ chocolate bar with $kj < nm$, that $kj - 1$ breaks are needed to separate the chocolate bar into all its squares.

We first break the chocolate bar in one place. Without loss of generality, suppose we break below the k^{th} row, where $1 \leq k \leq n - 1$, leaving us with two pieces, one of dimension $k \times m$ and the other of dimension $n - k \times m$. Note that by the strong induction hypothesis, we need break these two pieces $km - 1$ and $(n - k)m - 1$ times, respectively, in order to separate the chocolate bar into all its squares.

Therefore, the total number of breaks needed is

$$1+km-1+(n-k)m-1 = 1+km-1+nm-km-1 = nm-1.$$

Thus, by strong induction, the claim holds for any n, m.