21-127 Exam 1 Review Materials

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Major topics for your first exam include:

- Fluency in the notation and structure of propositional formulae, including building truth tables to establish logical equivalence and translating mathematical statements into propositional formulae
- Fluency in propositional operators, De Morgan's Laws for propositions
- Basics of proofwriting: how to structure proofs by direct proof, contradiction, contrapositive, cases, biconditional proofs. Be able to prove, if asked, that a given proof technique is valid.
- Understanding of quantifiers, when to use them, how to interpret them, and De Morgan's Laws for quantifiers
- Familiarity with the important number sets we use: $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$
- Peano Axioms and how we formalize numbers
- Induction, both strong and weak: you should know both how to use these tools, when to apply them, and why/how they work. You should be able to prove the Principles of Weak and Strong Induction from Peano Axioms.

Terms to know:

- Proposition
- Conjunction
- Disjunction
- Exclusive disjunction
- Negation
- Propositional Formula
- Tautology
- Conditional/Biconditional operator
- Universal/existential quantifier
- Limit
- Divides/divisible
- Prime

Some practice problems:

1. Prove that $(\neg(p \wedge q)) \wedge r$ is logically equivalent to $\neg((p \vee (\neg r)) \wedge (q \vee (\neg r)))$.

2. Prove that $(p \implies q) \wedge (\neg p \implies q)$ is logically equivalent to q.

3. Suppose that x and y both have the range of real numbers. Explain the difference between the following two statements.

   - $\forall x, \exists y, x^2 = y$
   - $\exists y, \forall x, x^2 = y$

4. Write the following statement as a propositional formula. Then prove it.

   Let $n \in \mathbb{N}$. If n^2 is divisible by 4 and n^3 is divisible by 27, then n is divisible by 6.

5. Write the following statement as a propositional formula. Then prove it.

   There is no integer value of x satisfying $0x = 1$.

6. Suppose $p(x)$ is a polynomial that can be written as

$$p(x) = (x - a_1)(x - a_2) \dots (x - a_n).$$

   Prove that α is a root of $p(x)$ if and only if $\alpha = a_i$ for some i with $1 \leq i \leq n$.

7. Let $a, u, b, v, d \in \mathbb{N}$. Recall the notation $d|a$ indicates that a is divisible by d. For each of the following, determine if it is true or false. Prove that your answer is correct.

   (a) If $d|a$ and $d|b$, then $d|(au + bv)$.

   (b) If $d|(au + bv)$ then $d|a$ and $d|b$.

   (c) If $d|a$ and d does not divide b or v, then d does not divide $au + bv$.

   (d) If d does not divide any of a, u, b, v, then d does not divide $au + bv$.

8. Find all real solutions to the equation

$$\sqrt{x + 10} + \sqrt{x + 5} = 5.$$

   Prove that your answer is correct.

9. Use the method of proof by contradiction to prove the AM-GM inequality: if $a, b \in \mathbb{R}$ with $a, b > 0$, then $\frac{a+b}{2} \geq \sqrt{ab}$.

10. Suppose that $a, b \in \mathbb{Z}$. Prove by contradiction that if $4|(a^2 - 3b^2)$ then at least one of a, b is even.

11. Prove, by contradiction, the following: $\forall x \in \mathbb{R}$, if $x \geq 1$, then $\sqrt{x} \leq x$.

12. On a certain island, each inhabitant always lies or always tells the truth. Calvin and Beatrice live on the island.

    Calvin says: "Exactly one of us is lying."

    Beatrice says: "Calvin is telling the truth."

    Determine who is telling the truth and who is lying. Prove that your answer is correct.

13. Let $n \in \mathbb{N}$ with $n \geq 2$. Suppose that for all $k \in \mathbb{N}$ with $2 \leq k \leq \sqrt{n}$, k does not divide n. Prove that n is prime.

14. Prove that for any $n \in \mathbb{N}$, $\displaystyle\sum_{k=0}^{n} k^3 = \left(\sum_{k=0}^{n} k\right)^2$.

15. Suppose $a_{n+1} = 5a_n - 6a_{n-1}$ for any $n \geq 1$, with $a_0 = 1$ and $a_1 = 1$. Prove that $a_n = 2^{n+1} - 3^n$ for all $n \in \mathbb{N}$.

16. Let $n \in \mathbb{N}$, with $n \geq 1$. Prove that $11^n - 6$ is divisible by 5.

17. Let $n \in \mathbb{N}$. Prove that $\displaystyle\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$.

18. Let $n, m \in \mathbb{N}$, with $n, m \geq 1$. A chocolate bar is made up of an $n \times m$ grid of squares. You break the chocolate bar into 1×1 pieces by iteratively breaking along a grid line. How many times must you make a break? Prove that your answer is correct.