21-127 Final Exam Practice Problems

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1. Use the Euclidean Algorithm to prove that for all $n \in \mathbb{N}$, the fraction $\frac{12n+1}{30n+2}$ is in lowest terms.

2. Use induction to prove that $\sum_{i=1}^{k}(2i-1) = k^2$.

3. Let $n \in \mathbb{N}$ be written, in base 10, as $111\cdots 11$, where there are 3^k 1s in the base expansion. Prove that n is divisible by 3^k.

4. Suppose you draw n straight lines in the plane, where no two lines are parallel and no three lines meet at a point. How many regions have you divided the plane into? Prove that your answer is correct.

5. Prove that $\sqrt{p}$ is irrational for any prime $p > 0$.

6. Define a sequence by $a_0 = 0$, $a_1 = 1$, and $a_n = 3a_{n-1} - 2a_{n-2}$ for $n \geq 2$. Derive and prove a nonrecursive formula for a_n.

7. Prove that for $x_1, x_2, \ldots, x_n \in \mathbb{R}$, with $x_i \geq 0$ for all i,
$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \ldots x_n}.$$

8. Let $k \in \mathbb{N}$, with $k \neq 0$. Prove that $k(k+1)(k+2)$ is divisible by 3.

9. Find the smallest $n > 0$ such that $n!$ is divisible by 990.

10. Let p, q be logical propositions. Prove that $(\neg(p \wedge q)) \wedge (p \vee q)$ is logically equivalent to $(p \wedge \neg q) \vee (q \wedge \neg p)$.

11. Let p and q be logical propositions.
    (a) Prove that $p \Rightarrow q$ is logically equivalent to $\neg q \Rightarrow \neg p$.
    (b) Explain in words why the above statement makes sense.

12. Let p, q, r be logical propositions. Prove that
$$[p \Rightarrow (q \Rightarrow r)] \Rightarrow [(p \Rightarrow q) \Rightarrow (p \Rightarrow r)]$$
is tautologically true (that is, it is true regardless of the truth values of p, q, r).

13. Suppose that $a, b, c, d \in \mathbb{N}$ with $ab - cd$ divides each of a, b, c, and d. Prove that $ab - cd = \pm 1$.

14. Suppose that $a \equiv b \pmod{n}$, and $c \equiv d \pmod{n}$. Prove that $ac \equiv bd \pmod{n}$.

15. Calculate the remainder of 3^{1000} when divided by 7.

16. (a) How many ways are there to rearrange the letters in the word "VECTOR"?

(b) How many ways are there to rearrange the letters in the word "TRUST," in such a way that the two Ts are not next to each other?

(c) How many ways are there to rearrange the letters in the word "ANALYSIS" so that no two consecutive letters are the same?

17. Let $n \in \mathbb{N}$, with $n \geq 1$. How many surjective functions are there from $[n+1]$ to $[n]$?

18. Prove that for all $n, m, k \in \mathbb{N}$, we have

$$\sum_{\ell=0}^{k} \binom{n}{\ell}\binom{m}{k-\ell} = \binom{n+m}{k}$$

19. Use counting in two ways to prove that for all $n, k \in \mathbb{N}$ with $n \geq k > 0$, we have

$$\sum_{j=n-k}^{n} \binom{n}{j} = \sum_{j=n-k}^{n} \binom{j-1}{n-k-1} 2^{n-j}.$$

20. Use Inclusion-Exclusion to determine the number of subsets of $[20]$ that contain a multiple of 5.

21. Let $U_1, U_2, \ldots, U_k$ be a finite partition of a set X, and let $A \subseteq X$. Prove that $A = \bigcup_{i=1}^{k}(A \cap U_i)$.

22. Let X and Y be sets. Prove that $X \subseteq Y$ if and only if $X = Y \backslash (Y \backslash X)$.

23. Define a function $f : \mathbb{R} \to \mathbb{R}$ by the following: for every $a \in \mathbb{R}$, let $f(a) = x$, where $x^2 + 2ax + a^2 = 0$. Prove that this function is well-defined.

24. Let X and Y be sets, and let $f : X \to Y$ be a function.

    (a) For $A, B \subseteq X$, prove that $f(A \cup B) = f(A) \cup f(B)$.

    (b) For $A, B \subseteq X$, prove that $f(A \cap B) \subseteq f(A) \cap f(B)$, but that sometimes these sets may not be equal.

25. Let $f : X \to Y$ be a function. Prove that f is injective if and only if $f(A \backslash B) = f(A) \backslash f(B)$ for every $A, B \subseteq X$.

26. Let $f : X \to X$ be a function, where X is a finite set. Prove that f is injective if and only if f is surjective. Explain why this is not true in the case that X is infinite.

27. We know that function composition is associative. Is it also commutative? Why/why not?

28. Suppose that A, B, C are countably infinite disjoint sets. Prove that $A \cup B \cup C$ is countably infinite directly, by finding a bijection between $A \cup B \cup C$ and $\mathbb{N}$.

29. For $1 \leq i \leq n$ define an interval $I_i = [a_i, b_i] \subseteq \mathbb{R}$, where $a_i < b_i$. Define a relation R on $[n]$ by iRj if and only if $b_i \leq a_j$ or $i = j$. Prove that R is a partial order.

30. Suppose that P is a poset that is not a lattice. Prove that P contains four elements, x, y, w, z such that $w \geq x$, $w \geq y$, $z \geq x$, $z \geq y$, but neither x and y nor w and z are comparable.

31. Suppose that (x_n) is a bounded real sequence. Prove that $\frac{x_n}{n^k}$ converges for all $k > 0$.

32. Find the limit of the sequence $x_n = \frac{n^3+n^2-1}{1-3n^3}$. Use tools from this class to prove that your answer is correct.

33. Prove that for all $x, y \in \mathbb{R}$, if $0 \leq x \leq y$, then $0 \leq \sqrt{x} \leq \sqrt{y}$.

34. Provide two proofs that the sequence of real numbers defined by $x_n = \frac{n+2}{n+3}$ for $n \in \mathbb{N}$ converges: one by definition and one using MCT.

35. Let $(\Omega, \mathbb{P})$ be a probability space. Suppose that $A \subseteq \Omega$ is independent from itself. What can you say about A?

36. Suppose we have a box containing a set of standard chess pieces. You select three pieces at random. Calculate each of the following probabilities.

    (a) $\mathbb{P}$(all three are the same color)

    (b) $\mathbb{P}$(all three are the same color | all three are pawns)

    (c) $\mathbb{P}$(all three are pawns | all three are the same color)

37. From a standard deck of cards, you deal out a five-card poker hand. What is the probability that there are at least three cards of the same value?

38. A couple does a genetic test, and finds that they are both carriers for the disease Cystic Fibrosis (CF). Each parent passes on the gene to a child with probability $\frac{1}{2}$. If a child has two copies of the gene (one from each parent), then that child will have the disease.

    (a) If the couple has n children, and we wish to count how many have CF, what distribution does this RV follow?

    (b) If the couple has 4 children, what is the expected number of children that have the gene for CF, but do not have the disease itself?

    (c) Suppose that the first 3 children the couple has do not carry the gene for CF. What can you infer about the 4th child?

39. We play the following game. First, we flip a fair coin. If it comes up heads, we roll a red die having the numbers 2, 4, 6, 8, 10, 12 painted on its sides. If it comes up tails, we roll a black die having the numbers 1, 2, 3, 4, 5, 6 painted on its sides. If the result is divisible by 4, you pay me \$4, and if the result is not divisible by 4, I pay you \$1.

    If we play this game n times, how much money should each player expect to have? Assume each player starts from \$0, and negative money is permitted.

40. Suppose that A is independent from B and B is independent from C. Does this imply that A is independent from C? If so, prove it. If not, find an example to show why not.

Sample Short Answer problems:

1. Explain, in your own words, how and why the Principle of Weak Induction works.

2. Let $p(x, y)$ be a logical formula. Explain the difference between the meaning of the statements $\forall x, \exists y, p(x, y)$ and $\exists y, \forall x, p(x, y)$, and how the proofs of these statements might also differ.

3. Explain, perhaps using pictures, why two sets are the same size if and only if there is a bijection between them.

4. State the definition of a convergent sequence. Negate that definition logically to obtain the definition of a divergent sequence. Discuss why this negation makes sense in space.

5. What is a random variable? Why might we define one?

6. State the definition of independent events in a probability space. Explain what this definition means.

7. Describe Cantor's diagonalization method (you need not fill in all details of the proof).