21-127 Exam 3 Review Materials

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Major topics for your third exam include:

- Infinite sets. Showing two sets are the same size via bijection.
- Countable sets: Showing a set is countable by listing, or by injection/surjection with another countable set. Products/unions of countable sets, and when they are also countable.
- Uncountable sets: Cantor's diagonalization argument.
- Relations: basic definitions, correspondence with subsets in $X \times Y$
- Equivalence relations: basic definitions, correspondence with partitions and functions
- Posets: basic definitions and examples (integers under divisibility, subsets under containment). Minimum, maximum, suprema, infima, lattices, distributive lattices. Do NOT worry about Boolean Algebras
- Fields: axioms, examples, basic properties. Ordered fields: same. Completeness in an ordered field: same. You can know (although it was not formally proved) that $\mathbb{R}$ is a complete ordered field.
- Magnitude in $\mathbb{R}$ and $\mathbb{R}^n$. Triangle Inequality. Cauchy-Schwarz Inequality.
- Convergence: definition. Understanding of how to prove a sequence is convergent/divergent. Basic theorems.
- Monotone Convergence Theorem.

Some practice problems:

1. Let X be a countable set. For each $n \in \mathbb{N}$, define $S_n = \{A \subseteq X \mid |A| = n\}$; that is, S_n is the set of all size n subsets of X. Let $S = \cup_{n\in\mathbb{N}} S_n$.
   (a) Prove that S is countable.
   (b) Is $S = \mathcal{P}(X)$? If so, prove it. If not, explain why not.
2. Let X be a set. Prove that X is at most countable if and only if there exists an injection $f : X \to \mathbb{N}$.
3. Let X be an infinite set, and let $S \subset X$ be finite. Prove that $|X| = |X \backslash S|$.
4. Let X and Y be sets. Suppose there exists an injection $f : X \to Y$ and a surjection $g : X \to Y$ be a surjection. Prove that there exists a bijection between X and Y.
5. Give an example of a relation that is neither symmetric nor antisymmetric.

6. Define a relation R on $\mathbb{R}$ by $x^2 R x$ for all $x \in \mathbb{R}$. Is this an equivalence relation? Explain.

7. Let $\Delta_X = \{(x, x) \in X \times X \mid x \in X\}$. This is called the diagonal subset of X. Prove that if R is an equivalence relation, then $\Delta_X \subseteq Gr(R)$.

8. Define a relation R on $\mathbb{N} \times \mathbb{N}$ by aRb if and only if $a \perp b$. Is this an equivalence relation? Explain.

9. Let $\sim$ be an equivalence relation on a set X. Prove that $x \sim y$ if and only if $[x]_\sim = [y]_\sim$.

10. Prove, using proerties of posets, that for any $a, b, c \in \mathbb{N}$,

$$\gcd(a, \gcd(b, c)) = \gcd(\gcd(a, b), c).$$

11. Let L be a distributive lattice, and let $x \in L$. Suppose that there exists a complement y to x in L. Prove that this complement is unique; that is, if y' is a complement to x, then $y = y'$.

12. Define an ordering $\preceq$ on $\mathbb{N} \times \mathbb{N}$ by the following:

    - If $a < b$, then $(a, c) \preceq (b, d)$ for all $c, d \in \mathbb{N}$.
    - If $a = b$, and $c \leq d$, then $(a, c) \preceq (b, d)$.

    This is called lexicographical ordering, and it works basically just like alphabetizing: first you look at the first letter. If the first letter is the same, then, you look at the second.

    (a) Prove that $\preceq$ is a partial order on $\mathbb{N} \times \mathbb{N}$.

    (b) Is $\preceq$ a lattice? If so, prove it. If not, give an explicit example of a set of elements that has either no supremum or no infimum.

    (c) If, in part (b), you found that $\preceq$ is a lattice, is it distributive? If so, prove it. If not, give an explicit example of a set of elements for which the distributive property fails.

13. Define an addition and multiplication on $\mathbb{R}^2$ as follows:

    - For $(a, b), (c, d) \in \mathbb{R}^2$, define $(a, b) + (c, d) = (a + c, b + d)$.
    - For $(a, b), (c, d) \in \mathbb{R}^2$, define $(a, b) \cdot (c, d) = (ac - bd, ad + bc)$.

    Prove that under this definition of addition and multiplication, $\mathbb{R}^2$ is a field. Determine an appropriate choice of 0 and 1 for the field.

14. Prove that there is no ordering $\leq$ on $\mathbb{C}$ under which $\mathbb{C}$ is an ordered field.

15. Prove that for $x_1, x_2, \ldots, x_n \in \mathbb{R}$,

$$\left| \sum_{i=1}^{n} x_i \right| \leq \sum_{i=1}^{n} |x_i|.$$

    When can we replace the inequality with equality?

16. Let $x_n = 2^n$ for all n. Prove that (x_n) does not converge.

17. Suppose that (x_n) is a sequence that does not converge, and that there exists some $M \in \mathbb{R}$ such that $|x_n| \leq M$ for all n. Prove that there exist two distinct points a, b such that for all $\epsilon > 0$ and for all $N \in \mathbb{N}$, there exists $n_1, n_2 > N$ such that $|x_{n_1} - a| < \epsilon$ and $|x_{n_2} - b| < \epsilon$.

18. Let $x_n = \frac{n^2}{(n-1)(n+1)}$. Prove that (x_n) converges.

19. Let $k \in \mathbb{N}$. Prove that the sequence $x_n = n^{-k}$ converges. (Hint: it'll go quickly if you use MCT)

20. Prove that the sequence $x_n = \frac{n^2+3n-5}{n^3+2n^2+1}$ converges. (Hint: I recommend using the previous problem, and maybe also the squeeze theorem)

21. (Challenge problem!) Let (x_n) be a sequence of distinct real numbers; that is, $x_n \neq x_m$ for any $n \neq m$. Suppose that the sequence is bounded; that is, there exists M such that $|x_n| \leq M$ for all n. Prove that there exists a sequence (y_n) such that

    - (y_n) is monotonic
    - For each n, $y_n = x_k$ for some k (that is, each y_n is one of the original x_k points)
    - If $y_n = x_k$ and $y_{n+1} = x_j$, then $k < j$ (that is, the sequence goes through the x points in order)