21-127 Exam 2 Review Materials

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Major topics for your second exam include:

- Basics of set theory. Including: understanding of unions, intersections, power sets, Cartesian Products, and how to prove two sets are equal.
- DeMorgan's Laws for sets.
- Basics of functions. Including: definition of function as well as key terms (domain, codomain, graph, well-defined, composition, image, preimage). Understanding of how to prove that a function is well-defined, and manipulate other basic properties.
- More sophisticated function stuff: understanding of injectivity, surjectivity, bijectivity. Understanding of how to prove a function satisfies any of these criteria. Knowing how these properties relate to left-, right-, and two-side inverses.
- Finite counting: how to use bijections to count finite sets. Inclusion/Exclusion. Counting in 2 ways. Basic theorems about counting unions, intersections, products, etc.
- GCDs. Know the definition, basic properties, Euclidean algorithm, Bezout's Lemma.
- Modular arithmetic. Know the basic definitions of what it means to be equivalent under a modulus, and simple definitions.
- Modular inverses: know when multiplicative inverses exist and how to find them (using, for example, Fermat's Little Theorem).
- Order of a number under modulus: Euler's Theorem.

Some practice problems:

1. Let A, B, C be subsets of the universe Z. Prove each of the following identities:
   (a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
   (b) $(A \cap B) \cup (A \cap B^c) = A$
   (c) $(A^c \cap Z)^c = A$
   (d) $A \times (B \cap C) = (A \times B) \cap (A \times C)$
2. Let $f(x) = x^2 - 3$ and let $g(x) = x^2 - 2$. Let $A = f(\mathbb{R})$ and let $B = g(\mathbb{R})$, the images of $\mathbb{R}$ under these functions. Prove that $B \subseteq A$ but $A \neq B$.
3. Let $A = \{x \in \mathbb{R} \mid -3 < x < 2\}$ and let $B = \{x \in \mathbb{R} \mid x^2 + x - 6 < 0\}$. Prove that $A = B$.

4. Let $f : X \to Y$ be a function. Suppose there exists $y_1, y_2 \in Y$, with $y_1 \neq y_2$, such that $f^{-1}(y_1) = f^{-1}(y_2)$. Prove that $y_1 \notin f(X)$ and $y_2 \notin f(X)$.

5. Let $\mathbb{Q}^+$ be the set of positive rational numbers. Let $f : \mathbb{Q}^+ \to \mathbb{N}$ be defined by $f(q) = n$, where n is the numerator of q. Is this a well-defined function? If so, prove it. If not, explain why not, and make an appropriate modification to produce a well-defined function.

6. Let $f : X \to Y$ be a function, and let $A \subseteq X$. Define $f|_A : A \to Y$ to be the function with $f|_A(a) = f(a)$ for all $a \in A$ (this is called the restriction of f to A. Prove that for all $B \subseteq Y$, $f|_A^{-1}(B) = f^{-1}(B) \cap A$.

7. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function defined by $f(x, y) = x$, and let $g : \mathbb{R} \to \mathbb{R}^2$ be a function defined by $g(x) = (x, 0)$. Show that g is a right inverse for f, but not a left inverse. Show that g is not unique as a right inverse; that is, there exists another function h that is a right inverse to f and $h \neq g$.

8. Prove that if f and g are both bijective, then $f \circ g$ is bijective, and $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

9. Let $f : X \to Y$ be a function with X and Y finite. Prove the following:

   (a) f is injective if and only if $|f^{-1}(A)| \leq |A|$ for all $A \subseteq Y$.

   (b) f is surjective if and only if $|f^{-1}(A)| > 0$ for all nonempty $A \subseteq Y$.

   (c) f is bijective if and only if $|f^{-1}(A)| = |A|$ for all $A \subseteq Y$.

10. Given $n \in \mathbb{N}$, define $D_n \subseteq \mathbb{Z}$ to be the set of divisors of n.

    Prove that for all $a, b \in \mathbb{N}$, $D_a \cap D_b = D_{\gcd(a,b)}$.

11. Use the Euclidean Algorithm to find $\gcd(660, 546)$.

12. For what values of k does the equation $2a + 4b = k$ have solutions? How do you know your answer is correct?

13. Let $a, b, c \in \mathbb{Z}$. Suppose $a \perp b$ and $a|bc$. Prove that $a|c$.

14. Prove that there does not exist $n \in \mathbb{Z}$ having $n \equiv 3 \pmod 4$ and $n \equiv 5 \pmod 8$.

15. Let $a, b \in \mathbb{Z}$, and n a modulus, with $a \equiv b \pmod n$. Prove that for all positive integers d such that $d|a$ and $d|b$, we have

$$\frac{a}{d} \equiv \frac{b}{d} \pmod{\frac{n}{\gcd(n, d)}}.$$

16. Let $n \geq 6$. Determine the remainder when $1! + 2! + 3! + \cdots + n!$ is divided by 9.

17. Let p and q be distinct positive primes. Prove that $p^q + q^p \equiv p + q \pmod{pq}$.

18. Let x be a positive three digit integer, written as $d_2 d_1 d_0$ (that is, these are the three digits of x. Let $b = x - d_2 * 100$; that is, b is just the second two digits of x. Prove that x is divisible by 7 if and only if $5d_2 - b$ is divisible by 7.

    Use this rule to determine if each of the following numbers are divisible by 7: 601, 521, 316, 686.

19. Suppose $X_1, X_2, \ldots, X_n$ are finite sets such that $X_i \cap X_j = \emptyset$ for all $i \neq j$. Prove, using Inclusion-Exclusion, that $|\cup_{i=1}^n X_i| = \sum_{i=1}^n |X_i|$.

20. Determine the number of positive integers less than 2000 that are multiples of $4, 7$, and 13.

21. You are playing a game of poker in which each player is dealt 5 cards, and then you bet, without being permitted to draw any more cards. You are dealt 4 10s and a K. How many hands are there that beat your hand?

    For reference, the only hands that beat 4-of-a-kind are higher 4-of-a-kind or straight flushes.

22. Let $n, m \in \mathbb{N}$. How many functions $f : [n] \to [m]$ are injections?

23. Prove, by counting in two ways, that for any $n, s \in \mathbb{N}$, we have

$$\sum_{k=1}^{n} \binom{n}{n-k}\binom{s-1}{k-1} = \binom{n+s-1}{n-1}.$$

24. Let $n \in \mathbb{N}$. Prove that

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$