21-127 Exam 2 Review Materials

Mary Radcliffe

Major topics for your second exam include:

- Basics of set theory. Including: understanding of unions, intersections, power sets, Cartesian Products, and how to prove two sets are equal.
- DeMorgan’s Laws for sets.
- Basics of functions. Including: definition of function as well as key terms (domain, codomain, graph, well-defined, composition, image, preimage). Understanding of how to prove that a function is well-defined, and manipulate other basic properties.
- More sophisticated function stuff: understanding of injectivity, surjectivity, bijectivity. Understanding of how to prove a function satisfies any of these criteria. Knowing how these properties relate to left-, right-, and two-side inverses.
- Finite counting: how to use bijections to count finite sets. Inclusion/Exclusion. Counting in 2 ways. Basic theorems about counting unions, intersections, products, etc.
- GCDs. Know the definition, basic properties, Euclidean algorithm, Bezout’s Lemma.
- Modular arithmetic. Know the basic definitions of what it means to be equivalent under a modulus, and simple definitions.
- Modular inverses: know when multiplicative inverses exist and how to find them (using, for example, Fermat’s Little Theorem).
- Order of a number under modulus: Euler’s Theorem.

Some practice problems:

1. Let $A, B, C$ be subsets of the universe $Z$. Prove each of the following identities:
   
   (a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
   (b) $(A \cap B) \cup (A \cap B^c) = A$
   (c) $(A^c \cap Z)^c = A$
   (d) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

2. Let $f(x) = x^2 - 3$ and let $g(x) = x^2 - 2$. Let $A = \{ x \in \mathbb{R} \}$ and let $B = g(\mathbb{R})$, the images of $\mathbb{R}$ under these functions. Prove that $B \subseteq A$ but $A \neq B$.

3. Let $A = \{ x \in \mathbb{R} \mid -3 < x < 2 \}$ and let $B = \{ x \in \mathbb{R} \mid x^2 + x - 6 < 0 \}$. Prove that $A = B$. 

4. Let \( f : X \to Y \) be a function. Suppose there exists \( y_1, y_2 \in Y \), with \( y_1 \neq y_2 \), such that \( f^{-1}(y_1) = f^{-1}(y_2) \). Prove that \( y_1 \notin f(X) \) and \( y_2 \notin f(X) \).

5. Let \( \mathbb{Q}^+ \) be the set of positive rational numbers. Let \( f : \mathbb{Q}^+ \to \mathbb{N} \) be defined by \( f(q) = n \), where \( n \) is the numerator of \( q \). Is this a well-defined function? If so, prove it. If not, explain why not, and make an appropriate modification to produce a well-defined function.

6. Let \( f : X \to Y \) be a function, and let \( A \subseteq X \). Define \( f|_A : A \to Y \) to be the function with \( f|_A(a) = f(a) \) for all \( a \in A \) (this is called the restriction of \( f \) to \( A \)). Prove that for all \( B \subseteq Y \), \( f|_A^{-1}(B) = f^{-1}(B) \cap A \).

7. Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be a function defined by \( f(x, y) = x \), and let \( g : \mathbb{R} \to \mathbb{R}^2 \) be a function defined by \( g(x) = (x, 0) \). Show that \( g \) is a right inverse for \( f \), but not a left inverse. Show that it is not unique as a right inverse; that is, there exists another function \( h \) that is a right inverse to \( f \) and \( h \neq g \).

8. Prove that if \( f \) and \( g \) are both bijective, then \( f \circ g \) is bijective, and \((f \circ g)^{-1} = g^{-1} \circ f^{-1}\).

9. Let \( f : X \to Y \) be a function with \( X \) and \( Y \) finite. Prove the following:
   
   (a) \( f \) is injective if and only if \( |f^{-1}(A)| \leq |A| \) for all \( A \subseteq Y \).
   
   (b) \( f \) is surjective if and only if \( |f^{-1}(A)| > 0 \) for all nonempty \( A \subseteq Y \).
   
   (c) \( f \) is bijective if and only if \( |f^{-1}(A)| = |A| \) for all \( A \subseteq Y \).

10. Given \( n \in \mathbb{N} \), define \( D_n \subseteq \mathbb{Z} \) to be the set of divisors of \( n \).

    Prove that for all \( a, b \in \mathbb{N} \), \( D_a \cap D_b = D_{\gcd(a,b)} \).

11. Use the Euclidean Algorithm to find \( \gcd(660, 546) \).

12. For what values of \( k \) does the equation \( 2a + 4b = k \) have solutions? How do you know your answer is correct?

13. Let \( a, b, c \in \mathbb{Z} \). Suppose \( a \perp b \) and \( a|bc \). Prove that \( a|c \).

14. Prove that there does not exist \( n \in \mathbb{Z} \) having \( n \equiv 3 \pmod{4} \) and \( n \equiv 5 \pmod{8} \).

15. Let \( a, b \in \mathbb{Z} \), and \( n \) a modulus, with \( a \equiv b \pmod{n} \). Prove that for all positive integers \( d \) such that \( d|a \) and \( d|b \), we have

\[
\frac{a}{d} \equiv b \pmod{\frac{n}{\gcd(n, d)}}.
\]

16. Let \( n \geq 6 \). Determine the remainder when \( 1! + 2! + 3! + \cdots + n! \) is divided by 9.

17. Let \( p \) and \( q \) be distinct positive primes. Prove that \( p^a + q^b \equiv p + q \pmod{pq} \).

18. Let \( x \) be a positive three digit integer, written as \( d_2d_1d_0 \) (that is, these are the three digits of \( x \). Let \( b = x - d_2 \times 100 \); that is, \( b \) is just the second two digits of \( x \). Prove that \( x \) is divisible by 7 if and only if \( 5d_2 - b \) is divisible by 7.

Use this rule to determine if each of the following numbers is divisible by 7: 601, 521, 316, 686.

19. Suppose \( X_1, X_2, \ldots, X_n \) are finite sets such that \( X_i \cap X_j = \emptyset \) for all \( i \neq j \).

Prove, using Inclusion-Exclusion, that \( |\bigcup_{i=1}^n X_i| = \sum_{i=1}^n |X_i| \).
20. Determine the number of positive integers less than 2000 that are multiples of 4, 7, and 13.

21. You are playing a game of poker in which each player is dealt 5 cards, and then you bet, without being permitted to draw any more cards. You are dealt 4 10s and a K. How many hands are there that beat your hand?
   For reference, the only hands that beat 4-of-a-kind are higher 4-of-a-kind or straight flushes.

22. Let \( n, m \in \mathbb{N} \). How many functions \( f : [n] \to [m] \) are injections?

23. Prove, by counting in two ways, that for any \( n, s \in \mathbb{N} \), we have
   \[
   \sum_{k=1}^{n} \binom{n}{n-k} \binom{s-1}{k-1} = \binom{n+s-1}{n-1}.
   \]

24. Let \( n \in \mathbb{N} \). Prove that
   \[
   \sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.
   \]