21-105 Pre-Calculus
Notes for Week # 3

Higher-Degree Polynomial Equations.

At this point we have seen complete methods for solving linear and quadratic equations. For higher-degree equations, the question becomes more complicated: cubic and quartic equations can be solved by similar formulas, and this has been known since the 16th Century: del Ferro, Cardan, and Tartaglia are all credited with having discovered the cubic equation, and Ferrari with the quartic equation. (Interestingly enough, the mainly self-taught genius Ramanujan, perhaps the 20th Century's most impressive mathematical mind, discovered his own method for solving the quartic after having been shown how to solve the cubic.) The situation becomes more bleak for higher-degree equations: Abel showed, in the first half of the 19th Century, that fifth degree and higher equations do not have similar formulas. And while he didn't live to see it done, the same result was a natural consequence of Evariste Galois's work, a French mathematician who died in a duel before his 21st birthday. I will not discuss the cubic or quartic formulas, but we need not always resort to the big guns to solve special problems:

Example: Solve $x^3 - 1 = 0$.

Solution: Our only x term is an x^3, so we can peel-the-onion:

$$\begin{aligned} x^3 - 1 &= 0 \\ x^3 &= 1 \\ \sqrt[3]{x^3} &= \sqrt[3]{1} \\ x &= 1, \end{aligned}$$

so we see that $x = 1$ is a solution. Note that we were able to take the cubic root of both sides of the equation without resorting to adding in "$\pm$"s anywhere. This is because every real number has exactly one real cubic root, since the order of the root (3) is odd. ■

Example: Solve $x^4 - 2x^2 + 1 = 0$.

Solution: While this is a quartic equation that doesn't consist of only one power of x, we do note that every power of x is even, so we make a substitution: if we let $u = x^2$ and solve for u and get numeric values, then we can substitute x^2 into those equations to get values of x:

$$x^4 - 2x^2 + 1 = (x^2)^2 - 2(x^2) + 1 = u^2 - 2u + 1,$$

and this is a quadratic in u with $a = 1$, $b = -1$, $c = 1$, so by the quadratic formula,

$$\begin{aligned} u &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(1)}}{2(1)} \\ &= \frac{2 \pm \sqrt{4-4}}{2} \\ &= \frac{2 \pm \sqrt{0}}{2} \\ &= \frac{2}{2} \\ &= 1, \end{aligned}$$

so $u = 1$. Substituting $u = x^2$ back in, we have $x^2 = u = 1$, so x is a square root of 1, meaning $x = \pm 1$. ■

Notice that linear equations have at most 1 solution and quadratic equations have at most 2 solutions. This is true in general: nth-degree polynomial equations have at most n real solutions.

Functions.

While the next topic may seem a bit unrelated to polynomial equations, it isn't. We're going to discuss a more general way of looking at an equation but we need a few definitions and ideas first.

Definition: Let A and B be sets. A **function** f from A into B is a rule that maps every element of A to exactly one element of B. We call the set A, the set of values that the function takes in its **domain**. We call the set of all elements of B that are associated with elements of f's domain the range of f.

Unless stated otherwise, we will deal with functions that take in real numbers (though possibly not ALL real numbers) as input, and produce real numbers as output: i.e. the domain and range will both be sets of real numbers.

Example: Let $f(x) = x$, the **identity function** from $\mathbb{R}$ into $\mathbb{R}$. This notation means that our function associates the x we put in (the x in $f(x)$) with the value on the right-hand side of the equation (which in this case is still x). The domain of f is $\mathbb{R}$, because clearly we can put any real value x into the formula x. The range of f is $\mathbb{R}$ as well, because if we let z be any real number, then letting $x = z$, so x is in the domain of f, we have $f(x) = x = z$, so $f(x) = z$ and z is in the range by definition. ■

Example: Let $f(x) = x^2$ from $\mathbb{R}$ into $\mathbb{R}$. Notice that the formula for $f(x)$ is a polynomial in x: we call functions of this type **polynomial functions**. The domain of f is $\mathbb{R}$, because if we take any real number x, then $f(x) = x^2$ is another real number because multiplication is closed in the reals . The range of f, on the other hand, is not all of $\mathbb{R}$: for example, there is no real x so that $x^2 = -1$, so -1 is not in the range of f. In fact, the range of f is the set of all nonnegative reals, which we can denote $[0, \infty)$ using interval notation. ■

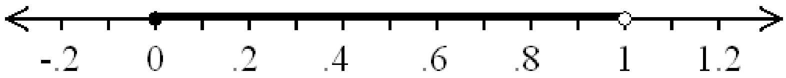
Interval Notation.
(Section 1.8 of JIT)

Let a and b be real numbers, with $a \leq b$. Then we define the following finite intervals from a to b:

$$\begin{aligned}
[a, b] &= \{x \mid a \leq x \leq b\}, \\
(a, b) &= \{x \mid a < x < b\}, \\
[a, b) &= \{x \mid a \leq x < b\}, \\
(a, b] &= \{x \mid a < x \leq b\},
\end{aligned}$$

so $[a, b]$ is the set of all numbers x satisfying $a \leq x \leq b$, etc. We call $[a, b]$ a **closed interval** (it contains both endpoints), (a, b) an **open interval** (it contains neither endpoint), and we refer to $[a, b)$ and $(a, b]$ as **half-open intervals**. We can think of intervals as being the connected part of the number line between a and b, and the use of parentheses or brackets determines which endpoints are also included.

Example: The interval $[0, 1)$ is the set of all numbers between 0 and 1, including 0 and excluding 1:



Additionally, with the use of $-\infty$ and ∞, we can define the following infinite intervals:

$$\begin{aligned} [a, \infty) &= \{x \mid a \leq x\}, \\ (a, \infty) &= \{x \mid a < x\}, \\ (-\infty, a] &= \{x \mid x \leq a\}, \text{ and} \\ (-\infty, a) &= \{x \mid x < a\}. \end{aligned}$$

Note that $-\infty$ and ∞ always have a parenthesis next to them, and never a bracket.

Example: Let $f(x) = 2x+5$. Determine the domain and range of f, as well as $f(0)$ and $f(3)$.

Solution: The domain of f is $\mathbb{R}$, because for every real number x, we have $2x + 5$ is also a real number.

The range of f is $\mathbb{R}$. Why? Suppose z is any real number. Then in order for z to be in the range of f, we must have an x in the domain ($\mathbb{R}$) with $f(x) = z$. Since $f(x) = 2x + 5$, we get that z is in the range of f if $2x + 5 = z$ has a real solution for x. But this is a linear equation in x, and solving for x we get that $x = \frac{z-5}{2}$, which is a real number and hence in the domain of f. Since z was any real number, and we showed z is in the range, that means the range must be every real number.

Finally, with substitution we see that

$$f(0) = 2(0) + 5 = 0 + 5 = 0$$

and

$$f(3) = 2(3) + 5 = 6 + 5 = 11. \blacksquare$$

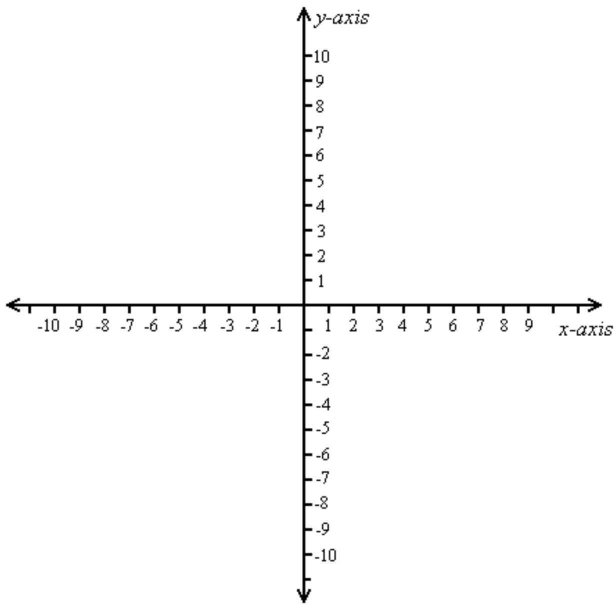
Example: Let $f(x) = \sqrt{x}$. Determine the domain and range of f.

Solution: Turning back to our definition of the radical symbol, we see that it specifically means the nonnegative square root of x. But we know that negative numbers do not have real square roots, so the domain is $[0, \infty)$.

Since $\sqrt{x} \geq 0$ for any nonnegative x, we expect the range to be $[0, \infty)$ as well: let z be a nonnegative real number. Then $z = \sqrt{z^2}$ (because $z \geq 0$), so if we take $x = z^2$, we have found an x with $f(x) = z$, so z is in the range, and the range is $[0, \infty)$.

The Cartesian Plane.

Definition: The perpendicular intersection of two number lines (one horizontal, one vertical), called the **coordinate axes**, at the 0 points forms an infinite plane, called the **Cartesian Plane**, where the horizontal axis (often called the x-axis) increases from left-to-right, and the vertical axis (often called the y-axis) increases from bottom-to-top. Every point in the plane is an **ordered pair** (x, y) of numbers, where x and y are the values along the x- and y-axis, respectively. The point $(0, 0)$, where the two axes intersect, is called the **origin**.



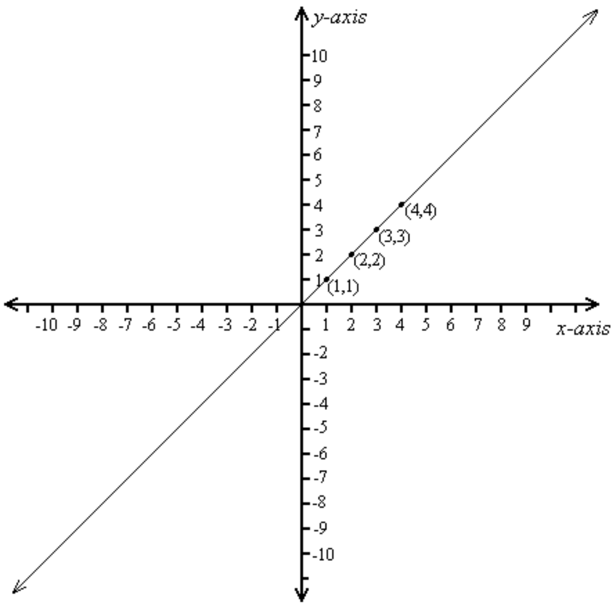
The Cartesian Plane

The Cartesian Plane was invented/discovered by Descartes in the early 17th Century, and was the foundation for analytic geometry, which allowed the ideas of Greek geometry and European algebra and analysis to combine. It's greatest use is in providing graphical representations of functions:

Definition: The **graph of a function** f is the set of points $\{(x, y)|y = f(x)\}$, and is traditionally shown by marking the points on the Cartesian Plane.

Example: Sketch the graph of $f(x) = x$.

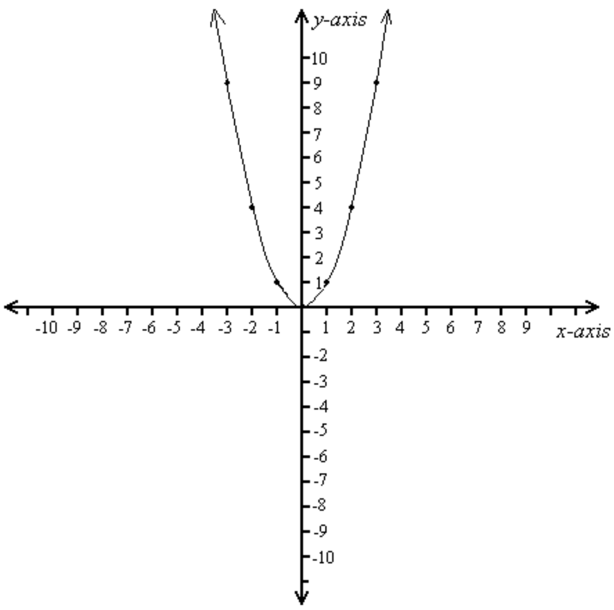
Solution:



Sketch of $f(x) = x$, with $(1, 1), (2, 2), (3, 3), (4, 4)$ marked.

Example: Sketch the graph of $f(x) = x^2$.

Solution:



Sketch of $f(x) = x^2$.

Lines.

We begin our serious look at functions with linear functions, whose graphs are lines. Lines are ubiquitous in calculus: in fact, differential calculus is primarily focused in using lines (linear functions) to approximate curves (nonlinear functions). Consequently, they warrant some special attention:

Definition: A **vertical line** is a set of all points satisfying the equation $x = c$ for some fixed c. Let $f(x) = mx + b$ be a linear function. We call the graph of f a **(non-vertical) line** in the plane. We call m the **slope** of the line, and b the **y-intercept**.

Example: Given two points in the plane $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, find an equation of the line that passes through both.

Solution: If $x_1 = x_2$, then the vertical line $x = x_1$ passes through them both, so we suppose $x_1 \neq x_2$. Then the line is the graph of $f(x) = mx + b$ for some m,b, so we need to find m and b with the conditions that $f(x_1) = y_1$ and $f(x_2) = y_2$ (so that P_1 and P_2 are in the graph of f). That gives us two linear equations,

$$mx_1 + b = y_1, \text{ and } mx_2 + b = y_2$$

in two unknowns (m, b). From the first equation, we get $b = y_1 - mx_1$, and substituting this into the second equation, we get

$$\begin{aligned}
mx_2 + b &= y_2 \\
mx_2 + (y_1 - mx_1) &= y_2 \\
mx_2 - mx_1 &= y_2 - y_1 \\
m(x_2 - x_1) &= y_2 - y_1 \\
m &= \frac{y_2 - y_1}{x_2 - x_1},
\end{aligned}$$

and since there is no m or b on the right-hand side, we have a definite value for m. Using our equation for b in terms of m, we see that

$$b = y_1 - mx_1 = y_1 - \left(\frac{y_2 - y_1}{x_2 - x_1}\right) x_1.$$

Therefore

$$f(x) = \left(\frac{y_2 - y_1}{x_2 - x_1}\right) x + y_1 - \left(\frac{y_2 - y_1}{x_2 - x_1}\right) x_1,$$

and since our line is the graph, its equation is therefore $y = f(x)$, or

$$y = \left(\frac{y_2 - y_1}{x_2 - x_1}\right) x + y_1 - \left(\frac{y_2 - y_1}{x_2 - x_1}\right) x_1. \blacksquare$$

From the example, we get two very important formulas: first, that the slope of a line through any two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ with $x_1 \neq x_2$ is

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

The slope is a measurement of the steepness that a function increases or decreases, which can be seen in the angles the graphed lines make with the x-axis. Moreover, we see that the slope of a vertical line would be undefined, since we would have to divide by 0.

The second formula we find is an equation for a line provided you know one point $P_1(x_1, y_1)$, and the slope:

$$y = mx + y_1 - mx_1,$$

or

$$y - y_1 = m(x - x_1),$$

the **point-slope form of a line**. The more standard form of writing the equation, $y = mx + b$, is called the **slope-intercept form**.

Example: Find the equation of a line passing through the point $(1, 1)$ with slope 7.

Solution: We're given the slope, $m = 7$, and an initial point $(x_1, y_1) = (1, 1)$, so the form of the equation to use is point-slope:

$$y - y_0 = m(x - x_0),$$

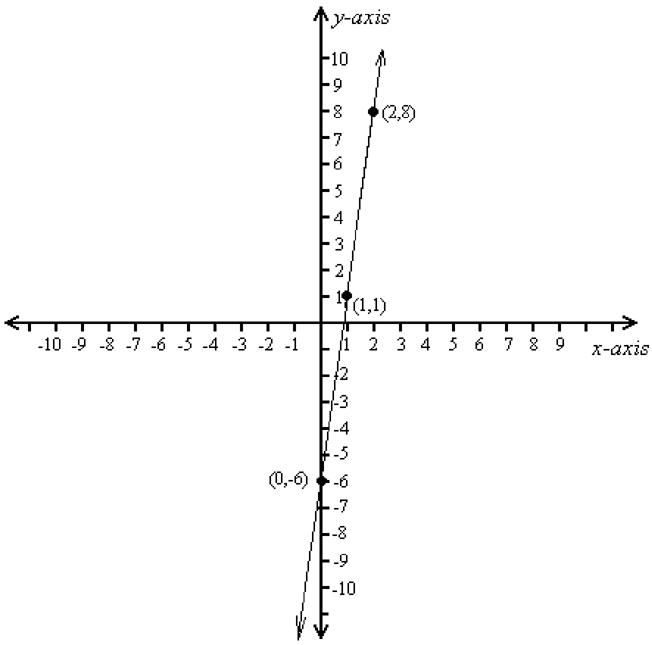
or

$$y - 1 = 7(x - 1).$$

Rewriting into slope intercept form by solving for y, we get

$$y = 7(x - 1) + 1 = 7x - 7 + 1 = 7x - 6,$$

so the line we want is $y = 7x - 6$:



Sketch of $y = 7x - 6$.

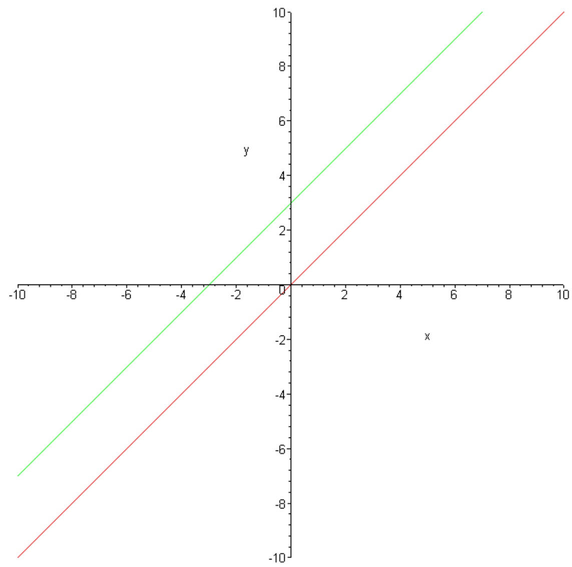
As was mentioned earlier, functions give us a slightly more general way of looking at equations:

Definition: We define the **graph of an equation** in x and y to be the set of all points (a, b) that satisfy the equation when a is substituted for x and b is substituted for y. From this definition, we see that the graph of a function $f(x)$ is really the graph of the *equation* $y = f(x)$.

So, when we say the graph of $f(x) = mx + b$, we also mean the graph of the equation $y = mx + b$. The two notations ($f(x)$ and y) will be used interchangeably.

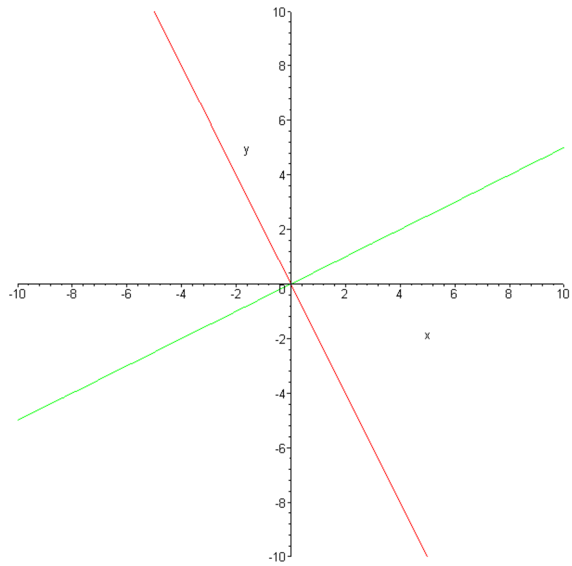
Definition: We say two (non-vertical) lines are **parallel** if they have the same slope. We say two lines are **perpendicular** if their slopes are negative reciprocals, i.e. the product of their slopes is -1.

Example: The lines $y = x$ and $y = x + 3$ are parallel: both have slope 1.



Sketch of $y = x$ and $y = x + 3$.

Example: The lines $y = \frac{1}{2}x$ and $y = -2x$ are perpendicular: the slope of $y = \frac{1}{2}x$ is $\frac{1}{2}$, the slope of $y = -2x$ is -2, and $(\frac{1}{2})(-2) = \frac{2}{-2} = -1$.



Sketch of $y = (1/2)x$ and $y = -2x$.

Let's look now at taking a function whose graph we know, say $y = f(x)$, and determining what function $g(x)$ gives us the same shape, but shifted up/down and left/right.

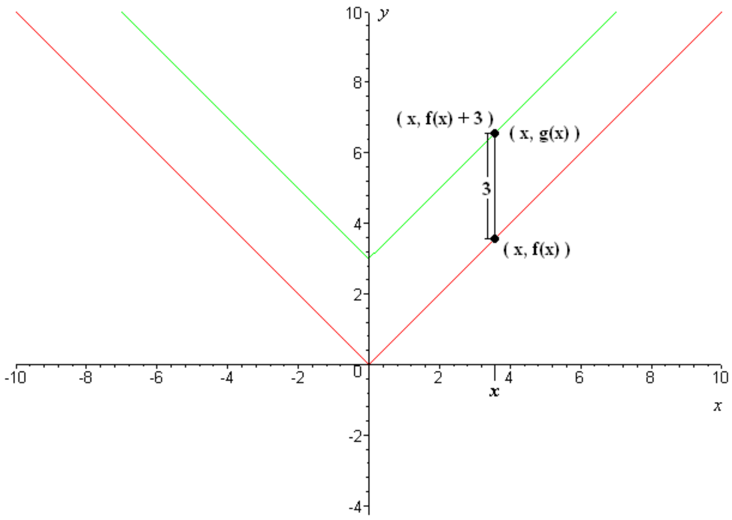
Shifting Graphs Up and Down:

(Section 4.4 in *JIT*.)

Example: Let $f(x) = |x|$, the **absolute value of** x, so

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

Suppose we look at the graph of f, and we want to shift the graph up by 3 units. Our goal will be finding the function $g(x)$ whose graph is precisely that.



Sketch of $y = |x|$ and $y = g(x)$.

If we take any value x in f's domain, and look at the corresponding point $(x, f(x))$ on the curve, we see that by going up 3 units, we get to the corresponding point on the graph of $g(x)$, $(x, g(x))$. But that point's y-coordinate is $f(x)+3$, so the point $(x, g(x))$ *is* the point $(x, f(x)+3)$. But for the points to be identical, their y-coordinates must be identical, so $g(x) = f(x)+3$, which, in this case means $g(x) = |x| + 3$, for all x in f's domain.

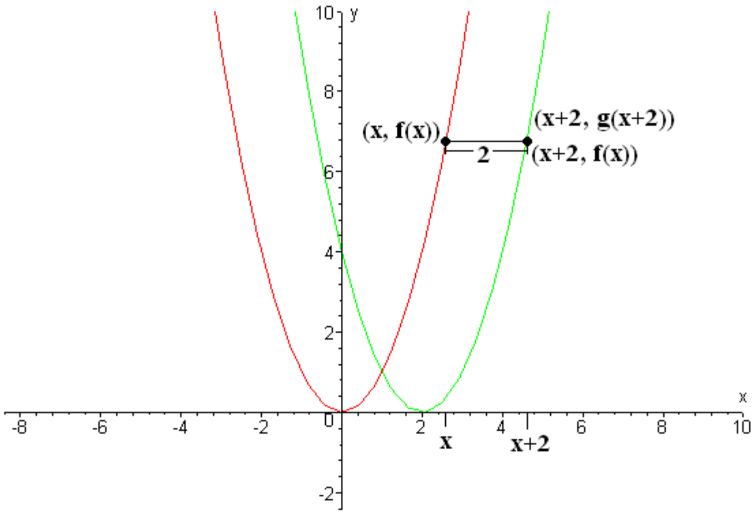
This idea works whether or not we used the number 3 or the number 5, or even the number -12, and is true for ANY function, not just $f(x) = |x|$:

- To shift the graph of a function $y = f(x)$ *up* by c units, you graph the function $y = f(x)+c$.

- To shift the graph of a function $y = f(x)$ *down* by c units, you graph the function $y = f(x) - c$.

Shifting Graphs Left and Right:
(Section 4.5 in *JIT*.)

Now, let's look at the graph of $f(x) = x^2$, and suppose we want to shift it to the right by 2 units to get the graph of a new function, $g(x)$:



Sketch of $y = x^2$ and $y = g(x)$.

Again, looking at a point $(x, f(x))$ on f's graph, we see that by going right 2 units (increasing the x-coordinate by 2), we get to the corresponding point on the curve of $g(x)$, $(x+2, g(x+2))$. But that point's y-coordinate value is $f(x)$, so $(x+2, g(x+2))$ is $(x+2, f(x))$, and this holds for any value of x. But, for the points to be identical, their y-coordinates must be identical, so $g(x+2) = f(x)$. This is true for any value of x, including $x-2$: $g((x-2)+2) = f(x-2)$, so $g(x) = f(x-2)$. As before, this pattern holds for all functions in a similar way:

• To shift the graph of a function $y = f(x)$ *left* by c units, you simply graph the function $y = f(x+c)$.

• To shift the graph of a function $y = f(x)$ *right* by c units, you simply graph the function $y = f(x-c)$.

Be careful! When shifting up or down by c we add or subtract c, but when shifting to the left we have to add c and shifting to the right we subtract. They're slightly different, so make sure you appreciate that!

Shifting Graphs both Up (or Down) and Right (or Left):
(Section 4.6 in *JIT*)

We can combine shifting up/down and left/right by simply doing one, then the other: to shift the graph of $f(x)$ up (or down) and then left (or right), just do the following:

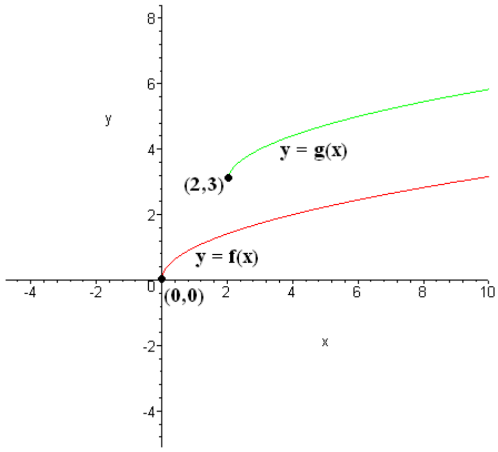
1. Take the graph of $y = f(x)$ and shift it up (or down), resulting in the graph of a new function, $y = h(x)$.

2. Shift the graph of that new function $h(x)$ left or right, resulting in the graph of yet another new function $y = g(x)$.

The graph of $y = g(x)$ is the solution. (In fact the order doesn't matter: we could shift left or right first, and then shift up or down. The end result will still be the same.)

Example: Suppose we shift the graph of $y = \sqrt{x}$ up 3 and to the right 2. Find the equation of the new graph:

Solution: Let $f(x) = \sqrt{x}$, so that we can just apply the 2-step process above: We first shift the graph up 3, which, by what we've done already, gives us the graph of the function $h(x) = f(x) + 3 = \sqrt{x} + 3$. Then, we shift the graph of $h(x)$ to the right by 2, which gives us the function $g(x) = h(x - 2) = \sqrt{x-2} + 3$. So the graph of $y = g(x)$, or $y = \sqrt{x-2} + 3$ is our desired solution:



Sketch of $y = f(x) = \sqrt{x}$ and $y = g(x)$. ■

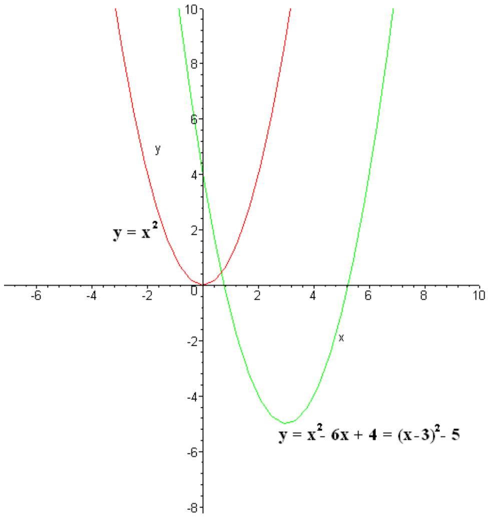
But now let's look at a slightly trickier problem:

Example: Graph the equation $y = x^2 - 6x + 4$.

Solution: Rather than simply plotting points and hoping we get the right answer, we can be a bit more clever: we can complete the square!

$$\begin{aligned}
y &= x^2 - 6x + 4 \\
&= (x^2 - 6x) + 4 \\
&= \left(x^2 + 2(-3)x\right) + 4 \\
&= \left(x^2 + 2(-3)x + (-3)^2 - (-3)^2\right) + 4 \\
&= \left(x^2 + 2(-3)x + (-3)^2\right) - (-3)^2 + 4 \\
&= (x + (-3))^2 - 9 + 4 \\
&= (x - 3)^2 - 5.
\end{aligned}$$

Why does this help us? Well, we've already seen how to graph $y = x^2$, and $y = (x-3)^2 - 5$ is the graph we'd get if we shifted $y = x^2$ down by 5, and then right by 3:



Sketch of $y = x^2$ and $y = x^2 - 6x + 4$. ■