21-105 Pre-Calculus
SAMS 2007
Notes for Week # 1

I have read over these notes a few times, but they are not perfect! If you find any mistakes, any typos, or come across something that doesn't make sense to you, I ask that you tell me about it! Everybody benefits from questions, especially me, so make sure you ask!

Pre-Calculus: What is it?

There are few courses so uniformly taught in the United States that are as poorly defined as Pre-Calculus, but the name does it the most justice: it is mathematics prior to Newton's discovery of the calculus, and that covers a bit of ground.

Looking only at Western mathematics, the development of the material we need can effectively begin with the Greeks: Euclid's *Elements*, published around 300 BC, is the first complete and rigorous treatment of geometry. Following Euclid, we also arrive at the greatest mathematician in history, Archimedes of Syracuse. His contributions to math and science are numerous and, in fact, understated, as he may have single-handedly developed the basics of integral calculus. His constant, π, the ratio of the circumference of a circle to its diameter, is where his impact will be most noticed in this course.

Archimedes was famously slain by a Roman soldier while drawing diagrams in the sand; the metaphor plays on, as the Roman Empire effectively stagnated mathematical development in the West for generations. The next major development came in the early 9th century from the Middle East, from a Arab scholar named Abu Ja'far Muhammad ibn Musa al-Khwarizmi. Al-Khwarizmi, in his book *Hisab al-jabr w'al-muqabala*, laid down the foundation for algebra (which comes from a transliteration of "al-jabr"), an extraordinary oversight of the Greeks given the direction their work had been heading in. But the West would still have to wait, as it wasn't until the 12th Century that his work would be translated into Latin and made accessible to Europeans.

Among those who would work on translating Arabic texts was Leonardo Fibonacci, whom you may have heard of from his interest in a particular sequence of numbers $(1, 1, 2, 3, 5, 8, 13, \ldots)$. Fibonacci, in his 1202 work *Liber Abace*, introduced the Hindu-Arabic positional number system to Europe, an innovation that certainly beat the alternative – Ever multiplied with Roman numerals? – and is still clearly in use today.

The next major step forward wouldn't come until the early 17th Century, when, in 1637, René Descartes would publish his *La Géométrie* and introduce what is now called analytic geometry, the idea of using coordinate systems to describe geometric objects, allowing classical problems in geometry (a discipline that was by now nearly 2000 years old) to be attacked by the new algebraic methods.

Descartes's work would open the door for calculus, in its infancy, to develop through the works of Fermat, Pascal, and Isaac Barrow, but it wouldn't be until 1666 that a student would make the major breakthrough. Isaac Newton, at home from school due to, among other things, the Great Fire of London, would discover the calculus by the age of 24. The discovery would be repeated, independently, 10 years later by a German mathematician named Gottfried Leibniz, and both men generally are given credit. However, Newton's works have been so influential and impressive that he is generally regarded as the third greatest mathematician in history.

Which means, from Greek geometry to the discovery of calculus was, roughly, two millennia of mathematical development, and we're going to learn it all (well, some of it) in a mere 6 weeks. You'll notice that I said that Archimedes was likely the greatest mathematician in history: his discovery of integral calculus could have advanced human understanding by 2000 years, had anyone actually *understood it.* As a result, I'm comfortable with placing him at the top of any list of great mathematicians.

And while we (probably) won't cover his work in our course, the second greatest mathematician in history is generally regarded as Carl Friedrich Gauss; the importance of his work to modern mathematics cannot be overstated. We will see other mathematicians as the course proceeds, but for now, let's turn our attention to some mathematics:

Sets. *Dun dun dunnnnnnn....*

Definition: A **set** is a collection of objects. That's all; it's just a bunch of things. We'll see them quite a bit, and we begin with sets of important numbers.

The first set of numbers are the **natural numbers**, which are often called the **counting** numbers:

$$\mathbb{N} = \{1, 2, 3, 4, 5, 6, \ldots\}.$$

This is standard set notation, so you'll see it in textbooks in future math courses. All it means is that $\mathbb{N}$ (our set) consists of the numbers 1, 2, 3, etc., and the ellipsis "..." is an easier way of writing "etc.". Whenever you see ellipses in math notation, it means that there should be stuff there that follows the pattern of what was immediately before or after, but writing it all out would take too long or would be impossible.

Next are the **integers**, which consist of the counting numbers, their opposites, and 0:

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}.$$

The **rational numbers** (or **fractions**) consist of fractions of integers, where the denominator is not 0. To write that as a set, we have:

$$\mathbb{Q} = \left\{ \frac{a}{b} \;\middle|\; a \in \mathbb{Z}, b \in \mathbb{N} \right\}.$$

This is slightly different set notation: instead of listing out all the things in our set in a row, we use $\{X|Y\}$, where X is a description of particular things in our set using variables, and Y are conditions on those things. The notation $\in$ means "is in", so $a \in \mathbb{Z}$ reads "a is in $\mathbb{Z}$", and it means that a is in the set $\mathbb{Z}$, and since $\mathbb{Z}$ is the set of integers, a is an integer.

Put together, our definition of $\mathbb{Q}$ reads " $\mathbb{Q}$ is the set of numbers of the form a/b, such that a is an integer and b is a natural number." Notice that we're not saying that fractions can't have negative denominators, but that all fractions can be written with positive denominators. For example,

$$\frac{1}{-2} = -\frac{1}{2} = \frac{-1}{2},$$

and -1 is an integer while 2 is a natural number.

To summarize our set notation so far, we look at another example: the set

$$\{x | x \in \mathbb{N} \text{ and } 1 \leq x \leq 10\}.$$

From what we've said, we read this set as "the set of all x such that x is in $\mathbb{N}$ *and* $1 \leq x \leq 10$." Since x is in $\mathbb{N}$ means x is a natural number, we see that this set could also be written as $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, or, using an ellipsis, $\{1, 2, 3, \ldots, 10\}$. We will see this set notation again, particularly when we study functions.

Finally, we're interested in the **real numbers**, $\mathbb{R}$, which are the numbers we're normally used to dealing with. Some examples are $0, 1, 3, 12, 4.8, \pi, e, \frac{1+\sqrt{5}}{2}, -1$. (That means all the numbers we've discussed so far are real numbers.)

To explain what the real numbers are, we consider decimal expansions: every real number has a decimal expansion. For some numbers, the total number of digits in their decimal expansion can be finite, like

$$1/2 = 0.5, \text{ or}$$

$$1/4 = 0.25,$$

and for some numbers it's infinite, like

$$1/3 = 0.333333\ldots$$

or

$$1/99 = 0.0101010101\ldots \; .$$

Some numbers even have infinitely many digits but no repeating pattern, like

$$\pi = 3.14159265358979323846264338327950288419716939937\ldots$$

and

$$e = 2.71828182845904523536028747135266249775724709369\ldots \; .$$

Numbers with infinitely long decimal expansions that don't have a repeating pattern are called **irrational numbers**. Why did we discuss decimal expansions? Because we can now define the real numbers as all numbers you can construct using decimals, which happen to be all of the rational numbers (like 1/2, 1/4, 1/3, and 1/99) and all the irrational numbers (like π and e).

Decimal notation, however, is not perfect: just as you can write fractions in two different ways, such as $\frac{1}{2} = \frac{2}{4}$, you can often write real numbers in two different ways. The most common example of this, which is *not* a trick, is

$$1 = 0.999999999\ldots$$

If you study calculus, particularly infinite series, you'll discover why, but we won't need that fact. It's just fun to know.

The real numbers (or just "reals") turn out to be the most important set of numbers above, and they are necessary for the calculus you will study to work properly, so we look at them more closely:

The Real Numbers.

We begin with the basic algebraic axioms of the reals: from these, we get most of the basic results of algebra. Don't let the name scare you: "algebraic" just means that it has to do with addition and multiplication, and an **axiom** is a statement that's assumed to be true with no justification given. Axioms are sometimes called "laws" or "properties", and for our purposes any of those words will do. These axioms will show exactly how real numbers are related to the operations of addition (+) and multiplication ($\cdot$). For what follows, assume that a, b, and c are any three real numbers.

1. Closure: $a + b$ and $a \cdot b$ are both real numbers: in other words, if you add two real numbers you get a real number, and if you multiply two real numbers you get a real number.

Example: Let $a = 1$ and $b = 2$. Then $a + b = 1 + 2 = 3$, and 3 is a real number, and $a \cdot b = 1 \cdot 2 = 2$, and 2 is a real number.

2. Commutativity: Order of addition and of multiplication do not matter:

$$a + b = b + a, \text{ and } a \cdot b = b \cdot a.$$

Example: Let $a = 1$ and $b = 2$ again: then $1 + 2 = 3$ and $2 + 1 = 3$, so $1 + 2 = 2 + 1$, and, similarly, $1 \cdot 2 = 2$ and $2 \cdot 1 = 2$, so $1 \cdot 2 = 2 \cdot 1$.

3. Associativity: The order of how you group terms (using parentheses) does not matter with addition and multiplication:

$$a + (b + c) = (a + b) + c, \text{ and } a \cdot (b \cdot c) = (a \cdot b) \cdot c.$$

Example: This time, let $a = 1$, $b = 2$, and $c = 36$: then $1 + (2 + 36) = 1 + 38 = 39$, and $(1 + 2) + 36 = 3 + 36 = 39$, so $1 + (2 + 36) = (1 + 2) + 36$. The same thing happens with multiplication: $1 \cdot (2 \cdot 36) = 1 \cdot 72 = 72$ and $(1 \cdot 2) \cdot 36 = 2 \cdot 36 = 72$.

4. Distributivity: Multiplication "distributes" over addition:

$$a \cdot (b + c) = a \cdot b + a \cdot c, \text{ and } (a + b) \cdot c = a \cdot c + b \cdot c.$$

Example: This time, let $a = 2$, $b = 4$, and $c = 6$: then $2 \cdot (4 + 6) = 2 \cdot 10 = 20$, and $2 \cdot 4 + 2 \cdot 6 = 8 + 12 = 20$, so $2 \cdot (4 + 6) = 2 \cdot 4 + 2 \cdot 6$. This is a very important property of numbers and will show up quite often!

5. Identities: There are unique real numbers 0 and 1 with the properties that

$$\begin{aligned} a + 0 = \quad & a \quad = 0 + a, \text{ and} \\ a \cdot 1 = \quad & a \quad = 1 \cdot a. \end{aligned}$$

The numbers 0 and 1 have to be defined for special reasons, but all we care about is that adding 0 to a number doesn't change it, and multiplying a number by 1 doesn't change it.

6. Inverses: For every real number a, there is a unique real number $-a$ such that

$$a + (-a) = 0 = (-a) + a.$$

$(-a)$ is called the **additive inverse** (or **opposite**) of a.

For every real number a, so that $a \neq 0$, there is a unique real number $\frac{1}{a}$ such that

$$a \cdot \left(\frac{1}{a}\right) = 1 = \left(\frac{1}{a}\right) \cdot a.$$

$\frac{1}{a}$ is called the **multiplicative inverse** (or **reciprocal**) of a.

Example: Let $a = 10$. Then $-a = -10$, and $1/a = 1/10 = 0.1$. Now, let $a = -3$: then $-a = -(-3) = 3$, and $1/a = 1/(-3) = -1/3$.

These six axioms let us justify most of the techniques you've may have already seen in algebra, and virtually everything we've already done. But to go any further, we also need subtraction and division:

Definition: Let a and b be real numbers. We define $a - b$ to be $a + (-b)$; this is called **subtraction**.

Definition: Let a and b be real numbers, and assume $b \neq 0$. Then $\frac{a}{b}$ is defined to be $a \cdot (\frac{1}{b})$; this is called **division**. We mention here that division by 0 is not defined, repeat, not defined!

Subtraction and division are exactly what you think they are; this is just a very specific way of looking at them. Next, let's look at some useful applications: multiplying and dividing fractions.

Multiplying and Dividing Fractions.

(Section 1.2 in *JIT*.)

We now use the axioms above to justify that

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}:$$

First, we need a preliminary result, which I'll call a proposition:

Proposition: If a and b are both real numbers and neither is 0, then $1/(a \cdot b) = (1/a) \cdot (1/b)$.

Proof: Consider $a \cdot b$. We know that there is only one number $1/(a \cdot b)$ whose product with $a \cdot b$ is 1, so we show that the right-hand side of our formula is that number:

$$\begin{aligned} a \cdot b \cdot \left(\tfrac{1}{a}\right) \cdot \left(\tfrac{1}{b}\right) &= a \cdot \left(\tfrac{1}{a}\right) \cdot b \cdot \left(\tfrac{1}{b}\right) && \text{(By the commutativity axiom.)} \\ &= (1) \cdot (1) && \text{(By the inverse axiom.)} \\ &= 1. && \text{(By the identity axiom.)} \end{aligned}$$

Since when we multiplied $a \cdot b$ by $(1/a) \cdot (1/b)$, we arrived at 1, that means that $(1/a) \cdot (1/b)$ is the reciprocal of $a \cdot b$, which is what we set out to show. ■

A proof is just an argument that justifies a claim I make. In mathematics, every statement that is either true or false must be shown to be true using a logical argument! **You will NOT be responsible for remembering or reproducing any proofs I give in class, but you will need to understand what the claims are.** Also, I will often use either shaded-in squares (■) or the letters Q.E.D. to denote the end of a proof. Q.E.D. means "quod erat demonstrandum", Latin for "that which was to be shown", meaning the argument has been completely presented. For our sake, it'll just mean "Quit, everything's done!".

Now, I'd like to justify that, for a, b, c, d all real numbers, $c, d \neq 0$, we have $(a/b) \cdot (c/d) = (a \cdot c)/(b \cdot d)$, using only our definitions above:

$$\begin{aligned} \frac{a}{b} \cdot \frac{c}{d} &= a \cdot \left(\frac{1}{b}\right) \cdot c \cdot \left(\frac{1}{d}\right) && \text{(By definition of division.)} \\ &= a \cdot c \cdot \left(\frac{1}{b}\right) \cdot \left(\frac{1}{d}\right) && \text{(By commutativity.)} \\ &= a \cdot c \cdot \left(\frac{1}{b \cdot d}\right) && \text{(By the Proposition.)} \\ &= \frac{a \cdot c}{b \cdot d}. \blacksquare && \text{(By definition of division again.)} \end{aligned}$$

Division is slightly (but not much!) more complicated:

$$\begin{aligned}
\left(\tfrac{a}{b}\right)\Big/\left(\tfrac{c}{d}\right) &= \left(\tfrac{a}{b}\right)\cdot\left(1\Big/\left(\tfrac{c}{d}\right)\right) && \text{(By definition of division.)}\\
&= \left(\frac{a}{b}\right)\cdot\left(1\Big/\left(c\cdot\frac{1}{d}\right)\right) && \text{(By definition of division.)}\\
&= \left(\frac{a}{b}\right)\cdot\left(\frac{1}{c}\cdot\left(\frac{1}{1/d}\right)\right) && \text{(By the Proposition.)}\\
&= \left(\frac{a}{b}\right)\cdot\left(\frac{1}{c}\cdot(d)\right) && \text{(By the inverse axiom.)}\\
&= \left(\frac{a}{b}\right)\cdot\left(d\cdot\left(\frac{1}{c}\right)\right) && \text{(By commutativity.)}\\
&= \left(\frac{a}{b}\right)\cdot\left(\frac{d}{c}\right) && \text{(By definition of division.)}\\
&= \frac{a\cdot d}{b\cdot c}. && \text{(By multiplication of fractions.)}
\end{aligned}$$

To summarize, when you multiply fractions you multiply the numerators and the denominators, and when you divide fractions, you multiply by the reciprocal of the divisor (what you're dividing by).

Example:

$$\left(\frac{2}{3}\right)\cdot\left(\frac{4}{5}\right)=\frac{2\cdot 4}{3\cdot 5}=\frac{8}{15},\ \text{and}\ \left(\frac{2}{3}\right)\Big/\left(\frac{4}{5}\right)=\left(\frac{2}{3}\right)\cdot\left(\frac{5}{4}\right)=\frac{2\cdot 5}{3\cdot 4}=\frac{10}{12}=\frac{5}{6}.$$

From this point on, I will use juxtaposition (writing numbers side-by-side) to denote multiplication, and stop writing the $\cdot$'s in unless I feel they're necessary (obviously $2\cdot 3$ and 23 are not the same number). So, when I write ab, I mean $a\cdot b$.

Adding and Subtracting Fractions.

(Section 1.3 in *JIT*.)

The trick to justifying addition and subtraction of fractions is a clever use of the number 1: we'll use it to construct the "cross-multiply" technique for addition: that

$$\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}.$$

To see this formula, we go as follows: (In each step I will only make changes with respect to a single axiom or proposition at a time, but I will not list which axioms justify the change. You should try to figure that out on your own.)

$$\begin{aligned}
\frac{a}{b}+\frac{c}{d} &= a\left(\frac{1}{b}\right)+c\left(\frac{1}{d}\right)\\
&= a\left(\frac{1}{b}\right)(1)+(1)c\left(\frac{1}{d}\right)\\
&= a\left(\frac{1}{b}\right)d\left(\frac{1}{d}\right)+b\left(\frac{1}{b}\right)c\left(\frac{1}{d}\right)\\
&= ad\left(\frac{1}{b}\right)\left(\frac{1}{d}\right)+bc\left(\frac{1}{b}\right)\left(\frac{1}{d}\right)\\
&= (ad+bc)\left(\frac{1}{b}\right)\left(\frac{1}{d}\right)\\
&= (ad+bc)\left(\frac{1}{bd}\right)=\frac{ad+bc}{bd}. \blacksquare
\end{aligned}$$

Subtracting fractions is just as easy, but to do so, we note two facts: that for every real number a, $-a=(-1)a$, and $a=\frac{a}{1}$. This way we can write opposites as products involving -1, which lets us do the following:

$$\begin{aligned}
\frac{a}{b}-\frac{c}{d} &= \frac{a}{b}+(-1)\left(\frac{c}{d}\right)\\
&= \frac{a}{b}+\left(\frac{(-1)}{1}\right)\left(\frac{c}{d}\right)\\
&= \frac{a}{b}+\left(\frac{(-1)c}{(1)d}\right)\\
&= \frac{a(1)d+b(-1)c}{b(1)d} \qquad \text{(From our addition formula from above!)}\\
&= \frac{ad+(-1)bc}{bd}\\
&= \frac{ad-bc}{bd}. \blacksquare
\end{aligned}$$

To summarize, we see that the addition of fractions is performed by a cross-multiplication technique, and subtraction is done similarly. This will become important when we come to adding and subtracting "rational" functions, which are fractions of polynomials.

Example:

$$\frac{2}{3}+\frac{4}{5}=\frac{2\cdot 5+3\cdot 4}{3\cdot 5}=\frac{10+12}{15}=\frac{22}{15}, \text{ and}$$
$$\frac{2}{3}-\frac{4}{5}=\frac{2\cdot 5-3\cdot 4}{3\cdot 5}=\frac{10-12}{15}=\frac{-2}{15}=-\frac{2}{15}.$$

Exponents.

(Section 1.4 in *JIT*.)

Definition: For any real number a and any natural number n, we define the following:

$$a^n=\underbrace{a\cdot a\cdot(\cdots)\cdot a}_{\text{n } a\text{'s}},$$
$$a^{-n}=\left(\frac{1}{a}\right)^n,$$

and

$$a^0 = 1.$$

In other words, a^n is the result of multiplying a by itself n times, a^{-n} is the result of dividing 1 by a exactly n times, and $a^0 = 1$.

Example:

$$2^3 = 2 \cdot 2 \cdot 2 = 8,$$

$$2^{-2} = \frac{1}{2^2} = \frac{1}{2 \cdot 2} = \frac{1}{4},$$

and, of course, $2^0 = 1$.

Laws of Exponents: Let m and n be integers, and let a and b be any real numbers. Then in the following, if the expressions on both sides exist, we have:

1. $a^m a^n = a^{m+n}$.
2. $\dfrac{a^m}{a^n} = a^{m-n}$.
3. $(a^m)^n = a^{m \cdot n}$.
4. $(a \cdot b)^n = a^n \cdot b^n$.
5. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$.
6. $\left(\dfrac{a}{b}\right)^{-n} = \left(\dfrac{b}{a}\right)^n$.

Roots (Radicals).

(Section 1.5 in *JIT*.)

Definition: Let x and y be real numbers, and let n be a natural number. We say that y is an n**th root** of x if $x = y^n$. We call n the **order** of the root. In the special cases $n = 2$ and $n = 3$, we say y is a **square** or **cube** root, respectively.

Example: 2 is a square root of 4, a cube root of 8, and a tenth root of 1024. The number -2 is also a square root of 4 and a tenth root of 1024, but it is not a cube root of 8. You should check that this is true!

When determining if a number has an nth root, we actually have two cases to consider: when n is an even number, and when it is an odd number:

Even Roots: Suppose n is even (divisible by 2):

• Every positive real number a has exactly two nth roots: we use **radical notation** $\sqrt[n]{a}$ to denote the **positive** root, and a's nth roots are $\sqrt[n]{a}$ and $-\sqrt[n]{a}$. In the case of square roots, we omit the "2" from the radical symbol.

• The number 0 has exactly one nth root, itself: $0 = \sqrt[n]{0}$.

• If $a < 0$, it does not have any real nth roots. If we discuss complex numbers, we'll see that negative numbers have imaginary nth roots, but no real ones.

Example: By our earlier example, we have $\sqrt{4} = 2$, $\sqrt[10]{1024} = 2$. However, as we observed, this means that 4 has two square roots, $\sqrt{4} = 2$ and $-\sqrt{4} = -2$. Similarly, 1024 has two tenth roots, $\sqrt[10]{1024}$ and $-\sqrt[10]{1024}$.

Odd Roots: Suppose now that n is odd:

• This case is much simpler, as every real number has exactly one nth root, which we again use radical notation to describe.

Example: $\sqrt[3]{8} = 2$ and $\sqrt[3]{-8} = -2$.

Radical notation, while standard, is a turn in the wrong direction, notationally, so we attempt to correct that:

Definition: For a natural number n and a real number a, we define $a^{1/n} = \sqrt[n]{a}$ (if it exists).

All we've just said is that, if we have $y = x^{1/n}$, we read that as "y is an n-th root of x", and it means $y^n = x$.

So now we can let our exponents be integers or reciprocals of positive integers. I mean, come on, what's left?

Definition: Let $r \in \mathbb{Q}$, so we can write $r = m/n$ where $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. If $a \in \mathbb{R}$ and $\sqrt[n]{a}$ exists, then we define

$$a^r = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m.$$

While they don't look the same, $\sqrt[n]{a^m}$ and $(\sqrt[n]{a})^m$ are the same number, if they exist.

Example: $(27)^{2/3} = (\sqrt[3]{27})^2 = (3)^2 = 9$.

In fact, with this definition (and with a good amount of work), we can show that the Laws of Exponents hold when we consider rational exponents instead of merely integers.

Laws of Exponents, Expanded: Let r and s be rational numbers, and let a and b be any real numbers. Then in the following, if the expressions on both sides exist, we have:

1. $a^r a^s = a^{r+s}$.
2. $\dfrac{a^r}{a^s} = a^{r-s}$.
3. $(a^r)^s = a^{r \cdot s}$.
4. $(a \cdot b)^r = a^r \cdot b^r$.
5. $\left(\dfrac{a}{b}\right)^r = \dfrac{a^r}{b^r}$.
6. $\left(\dfrac{a}{b}\right)^{-r} = \left(\dfrac{b}{a}\right)^r$.

Example:

$$\begin{aligned}(1024^{1/5}) \cdot (1024^{1/2}) &= 1024^{1/2+1/5} = 1024^{7/10} = \sqrt[10]{1024^7} = (\sqrt[10]{1024})^7 \\ &= (\sqrt[10]{2^{10}})^7 = (2^{10/10})^7 = (2^1)^7 = 2^{1 \cdot 7} = 2^7 = 128.\end{aligned}$$

Though it's terrible notation, from the rules for exponents we get the following "laws" for roots:

Root Laws: Let $n \in \mathbb{N}$ and let $a, b \in \mathbb{R}$. Then if the expressions on both sides exist, we have:

$$\begin{array}{ll} 1. & \sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b} \\ 2. & \sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}. \end{array}$$

Note: In each of the laws for exponents above, notice that additions and subtractions only occur in the exponent (with the r and s, not the a and b). In fact, most of the time,

$$(a+b)^r \neq a^r + b^r.$$

This means that, among other things,

$$\frac{1}{a+b} \neq \frac{1}{a} + \frac{1}{b}.$$

This error has become so common that some people refer to it as the "Freshman Rule". Rather than simply tell you that $(a+b)^r$ is more complicated, I've decided to, for the special case that r is a natural number, show you what $(a+b)^r$ really is.

The Binomial Theorem.

(Appendix E in *JIT.*)

Let $a, b \in \mathbb{R}$, $n \in \mathbb{N}$. As we've just seen, the laws for exponents let us break apart expressions of the form $(ab)^n$ into $a^n \cdot b^n$ (which sometimes can be easier to evaluate), but there is no similarly simple formula for $(a+b)^n$. We would like a formula that allows us to "simplify" the expansion, in such a way that we don't have to perform the multiplications and additions each time.

First, let's look at the first couple powers of $(a+b)$:

$$\begin{array}{lcl c c} (a+b)^0 & = & 1 & = & (1)a^0b^0 \\ (a+b)^1 & = & a+b & = & (1)a^1b^0 + (1)a^0b^1 \\ (a+b)^2 & = & a^2+2ab+b^2 & = & (1)a^2b^0 + (2)a^1b^1 + (1)a^0b^2 \\ (a+b)^3 & = & a^3+3a^2b+3ab^2+b^3 & = & (1)a^3b^0 + (3)a^2b^1 + (3)a^1b^2 + (1)a^0b^3 \end{array}$$

Notice, that on the far right-hand side, we've grouped the coefficients of each term in parentheses, and we've written in exponents for a and b for each term (the terms without a or b have a^0 or b^0 in their place).

Now, if we go from left to right, we notice that the powers of a start at n and decrease by 1 in each term until we get to 0, which we'll illustrate on $(a+b)^3$:

$$(1)a^{\mathbf{3}}b^0 + (3)a^{\mathbf{2}}b^1 + (3)a^{\mathbf{1}}b^2 + (1)a^{\mathbf{0}}b^3,$$

while at the same time the powers of b start at 0 and increment by 1 until we get to n:

$$(1)a^3b^{\mathbf{0}} + (3)a^2b^{\mathbf{1}} + (3)a^1b^{\mathbf{2}} + (1)a^0b^{\mathbf{3}}.$$

This is a fairly simple pattern, and it's one we should expect: when we expand $(a+b)^n$ out completely (without grouping or reordering terms), every term should be a product of n a's and b's:

$$\begin{aligned} (a+b)^n &= a(a+b)^{n-1} + b(a+b)^{n-1} \\ &= aa(a+b)^{n-2} + ab(a+b)^{n-2} + ba(a+b)^{n-2} + bb(a+b)^{n-2} \\ &= \cdots, \text{ and so on.} \end{aligned}$$

We expect to come across every term of the form $a^j b^k$ where j and k are nonnegative integers with $j+k=n$, so when we regroup, we can write the terms in the order I have, so that a's exponent decreases and b's exponent increases. This means that the only pattern we don't have is the pattern for the coefficients, but that pattern can be found in Pascal's Triangle (first studied by Yanghui in China in the 12th Century):

Pascal's Triangle: Pascal's Triangle is a number triangle such that every row starts and ends with 1, and all terms in between are the sum of the two numbers above them. For technical reasons, we say that it starts with the 0th row instead of the first, but you'll see why soon:

$$\begin{array}{ccccccccccccl}
 & & & & & 1 & & & & & & & \text{0th row} \\
 & & & & 1 & & 1 & & & & & & \text{1st row} \\
 & & & 1 & & 2 & & 1 & & & & & \text{2nd row} \\
 & & 1 & & 3 & & 3 & & 1 & & & & \text{3rd row} \\
 & 1 & & 4 & & 6 & & 4 & & 1 & & & \text{4th row} \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \text{5th row} \\
\iddots & & & & & & & & & & \ddots & & \vdots
\end{array}$$

Looking back at our powers of $(a+b)$, we see that our coefficients form the rows of the triangle such that the coefficients of $(a+b)^0$, $(a+b)^1$, $(a+b)^2$, and $(a+b)^3$ are the same numbers as in rows 0, 1, 2, and 3 respectively. This is not a coincidence, but is, in fact, true for all n:

<u>**The Binomial Theorem:**</u> Let a and b be real numbers, and let n be a non-negative integer. Then $(a+b)^n$, when expanded out completely, is

$$\binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n}a^{n-n}b^n,$$

where the numbers

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}$$

form the nth row of Pascal's Triangle.

The numbers $\binom{n}{k}$ (pronounced "n choose k") are called **binomial coefficients** for what now must be obvious reasons.

<u>**Example:**</u> Let's look at $(x+y)^5$: the 5th row of Pascal's Triangle is $1, 5, 10, 10, 5, 1$, so we use the formula we have

$$\begin{aligned}
(x+y)^5 &= (1)x^5y^0 + (5)x^4y^1 + (10)x^3y^2 + (10)x^2y^3 + (5)x^1y^4 + (1)x^0y^5 \\
&= x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5.
\end{aligned}$$

<u>**Example:**</u> Suppose now that $a=1$ and $b=1$, and $n=3$, so we have $(1+1)^3 = 2^3 = 8$. By the binomial theorem, since the third row is $1, 3, 3, 1$,

$$\begin{aligned}
(1+1)^3 &= (1)1^3 \cdot 1^0 + (3)1^2 \cdot 1^1 + (3)1^1 \cdot 1^2 + (1)1^0 \cdot 1^3 \\
&= 1 + 3 + 3 + 1 \\
&= 8. \blacksquare
\end{aligned}$$

Pascal's Triangle is an easy way to calculate the coefficients for small values of n, but we'd also like another formula for $\binom{n}{k}$, one that doesn't require us to reconstruct the triangle each time.

Fortunately, such a formula exists, but it requires (yet) another definition:

Definition: Let $n \in \mathbb{N}$. Then the **factorial** of n, denoted $n!$, is the number

$$n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1,$$

i.e. the product of the first n natural numbers. Additionally, we define $0! = 1$.

Examples: $2! = 2 \cdot 1 = 2, \quad 3! = 3 \cdot 2 \cdot 1 = 6, \quad 4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24.$

Claim: For $n \in \mathbb{N}$, and $k \in \mathbb{Z}$ with $0 \le k \le n$,

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Example:

$$\begin{aligned}
\binom{4}{0} &= \frac{4!}{0! \cdot (4-0)!} = \frac{4!}{1 \cdot 4!} = \mathbf{1}, \\
\binom{4}{1} &= \frac{4!}{1! \cdot (4-1)!} = \frac{24}{1 \cdot 3!} = \frac{24}{6} = \frac{24}{6} = \mathbf{4}, \\
\binom{4}{2} &= \frac{4!}{2! \cdot (4-2)!} = \frac{24}{2 \cdot 2!} = \frac{24}{4} = \mathbf{6}, \\
\binom{4}{3} &= \frac{4!}{3! \cdot (4-3)!} = \frac{24}{6 \cdot 1!} = \frac{24}{6} = \mathbf{4}, \\
\binom{4}{4} &= \frac{4!}{4! \cdot (4-4)!} = \frac{4!}{4! \cdot 0!} = \frac{4!}{4! \cdot 1} = \mathbf{1}.
\end{aligned}$$

Thus, checking above, we see that the numbers $\binom{4}{0}, \binom{4}{1}, \binom{4}{2}, \binom{4}{3}, \binom{4}{4}$ are the fourth row of Pascal's Triangle, so our formula for $\binom{n}{k}$ seems right. (It is.)

Polynomials and Polynomial Equations.

(Chapter 3 in *JIT*.)

Definition: Let x be a variable, let a be a constant with respect to x, and let n be a non-negative integer. Then a **monomial in** x is an expression of the form ax^n, where $x^0 = 1$. (This is exponentiation, i.e. x^n is x multiplied by itself n times.)

Example: $3x^2$, $12x^{15}$, and 1 are all monomials in x.

Definition: A **polynomial in** x is a sum of monomials in x, which can be written as

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_0 x^0,$$

where $a_n, a_{n-1}, \ldots, a_1, a_0$ are all constants with respect to x and are called the **coefficients of** f.

Example: $3x^2$, $3x^2 + 1$, and 0 are all polynomials in x:

$$3x^2 = 3x^2 + 0x^1 + 0x^0, 3x^2 + 1 = 3x^2 + 0x^1 + 1x^0, \text{ and}$$

$$0 = 0x^0.$$

Definition: Let $f = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x^1 + a_0 x^0$ be a polynomial. If $a_n \neq 0$, then we say that n is the **degree of** f. The degree of the zero polynomial $f = 0$ is undefined.

Example: Let $f = 3x^2 + 2x + 1$, $g = 3x + 2$, and $h = 1 + 2x + x^2$. Then the degree of f is 2, the degree of g is 1, and the degree of h is 2. (Why does h have degree 2? Because when we write the terms in decreasing order of powers of x, we get $h = x^2 + 2x + 1$, and the x^2 is the highest power of x we have left over, so the degree is 2.)

Earlier I stated that the coefficients had to be constants "with respect to x". What that really means is that they can be variables or contain variables, but that x cannot be one of those variables, i.e. the value of x does not change the value of the constants:

Example: $f = x^3y^4z^5$ is a polynomial in x of degree 3, a polynomial in y of degree 4, and a polynomial in z of degree 5.

Polynomials can be added and multiplied in exactly the same way that real numbers can. Moreover, for a fixed exponent n, we call all terms in the sum defining f that all have x^n **like terms**.

Example: Let $f = 3x^2 + 2x + 1$ and $g = 3x + 2$ be polynomials in x. Then

$$\begin{aligned} f + g &= (3x^2 + 2x + 1) + (3x + 2) \\ &= 3x^2 + 2x + 1 + 3x + 2 \\ &= 3x^2 + (2 + 3)x + (1 + 2) \\ &= 3x^2 + 5x + 3 \end{aligned}$$

is a polynomial in x, and

$$\begin{aligned} f \cdot g &= (3x^2 + 2x + 1) \cdot (3x + 2) \\ &= (3x^2 + 2x + 1)3x + (3x^2 + 2x + 1)2 \\ &= (3x^2)3x + (2x)3x + (1)3x + (3x^2)2 + (2x)2 + (1)2 \\ &= 9x^3 + 6x^2 + 3x + 6x^2 + 4x + 2 \\ &= 9x^3 + (6 + 6)x^2 + (3 + 4)x + 2 \\ &= 9x^3 + 12x^2 + 7x + 2 \end{aligned}$$

is a polynomial in x.

Definition: Certain polynomials have names based on their degrees:

Degree	**Name**
1	Linear
2	Quadratic
3	Cubic
4	Quartic
5	Quintic

Example: $f = 3x^2 + 2x + 1$ is a quadratic polynomial, but $g = 3x + 2$ is a linear polynomial.

Polynomials arise often in applications, but usually when they arise, they arise in the form of an equation, and the variable x represents an unknown quantity we want to solve for. So, we now consider polynomial equations:

Definition: A **polynomial equation in** x is an equation of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0,$$

where $a_n, a_{n-1}, \ldots, a_0$ and $b_m, b_{m-1}, \ldots, b_0$ are constants with respect to x. (In other words, a polynomial equation is an equation with polynomials on both sides.) If a is a real number, we call a a **solution** of the equation if by substituting $x = a$ into both sides, the equation holds (is true).

Example: $3x + 2 = 4x + 1$ is a polynomial equation in x.

1 (or we'll write $x = 1$) is a solution of the equation because $3(1) + 2 = 5$ and $4(1) + 1 = 5$, so by substituting $x = 1$ into both sides, the equation is true.

$x = 0$ is not a solution, since $3(0) + 2 = 2$ but $4(0) + 1 = 1$, and both sides are not the same.

The easiest type of polynomial equation to solve is a **linear equation**: when the maximum degree of either polynomial is 1, which will look like

$$rx + s = tx + u,$$

where r, s, t, u are constants. The first thing we note is that we can subtract the right-hand side from both sides and we'll have an equation with the exact same solutions:

$$rx + s - (tx + u) = 0,$$

or we can write it as

$$(r - t)x + (s - u) = 0.$$

Since $r - t$ is a number, and $s - u$ is a number, we see that solving linear equations depends on our ability to solve the linear equation

$$mx + b = 0,$$

where m and b are constants.

However, we can do that in a straightforward way from the properties we've already discussed. First, if $m = 0$, then our equation is really $(0)x + b = 0$, and the right-hand side is just b, so every value of x is a solution if $b = 0$, and no value of x is a solution if $b \neq 0$. Otherwise, $m \neq 0$, so we can do the following:

$$\begin{aligned} mx + b &= 0 \\ mx + b - b &= 0 - b \\ mx &= -b \\ \frac{mx}{m} &= \frac{-b}{m} \\ x &= \frac{-b}{m} = -\frac{b}{m}. \end{aligned}$$

So, in order for $mx + b = 0$, we must have $x = -b/m$, and that's the only solution! We will call writing linear equations in the form $mx + b = 0$ as the **standard form**.

Example: Solve $3x + 2 = 2x + 1$: I'll solve this two ways: the first is the most basic: get the terms with x on one side, and the terms without x on the other side:

$$\begin{aligned} 3x + 2 &= 2x + 1 \\ 3x + 2 - 2x &= 2x + 1 - 2x \\ x + 2 &= 1 \\ x + 2 - 2 &= 1 - 2 \\ x &= -1, \end{aligned}$$

and we see the solution is $x = -1$. JIT calls this method for solving equations "peeling-the-onion", and it's a very useful thing to do in general.

(Note that whatever we do to one side of the equation, we then do to the other! This is VERY important to keep in mind!)

Example: Solve $3x + 2 = 2x + 1$ again: Now, we solve the exact same equation by setting it into standard form and just use what we've already seen:

$$\begin{aligned} 3x + 2 &= 2x + 1 \\ 3x + 2 - (2x + 1) &= 2x + 1 - (2x + 1) \\ (3 - 2)x + (2 - 1) &= 0 \\ x + 1 &= 0. \end{aligned}$$

This is the standard form, where $m = 1$ and $b = 1$, so by our (general) solution above, that means the solution must be $x = -(1)/(1) = -1$, which we know is the answer!

So why did we solve the same problem twice, and why did we bother to set the equation into standard form the second time if the first method worked? Well, the reason is that the first method works very well for linear equations and for some very "nice" higher-degree equations, but the methods for solving higher-degree equations assume that they're in standard form to begin with, and, additionally, when we discuss functions (next), we'll see why standard form is a very "natural" way of writing equations.