Statement of Research Interests
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My current research interests are varied, but primarily focused on extremal and probabilistic combinatorics, especially problems in extremal graph theory and random graph theory.

Anti-Ramsey Theory

Classical Ramsey-theoretic questions on graphs involve coloring the edges of a graph with a fixed number of colors, say k, and determining whether a given monochromatic subgraph exists [13]. Anti-Ramsey questions, on the other hand, consider other restrictions on the colorings (for example, using each color at most b times [17]), and search for subgraphs in which no color is repeated [10].

An edge-coloring c of a graph G is *b-bounded* if no color occurs more than b times, and G is *rainbow* under c if $c(e) \neq c(f)$ for all distinct edges e, f of G. Consider the following question, posed by Bohman, Frieze, Pikhurko and Smyth in [3]: given a (finite) tree H, what is the minimum size of a tree T such that under every b-bounded coloring of T a rainbow copy of H exists? Writing $T \rightsquigarrow (H; b)$ for this condition, let

$$AR(H; b) = \min_{T \rightsquigarrow (H;b)} |E(T)|.$$

In [3], the authors showed that $AR(H; b) < \infty$ by constructing a class of (finite) trees $\mathcal{B}_{H,b}$ such that for all $T \in \mathcal{B}_{H,b}$, $T \rightsquigarrow (H; b)$. Moreover, they conjectured that the bound provided by these constructions is sharp.

Conjecture 1 (Bohman, Frieze, Pikhurko and Smyth [3])

$$AR(H; b) = \min_{T \in \mathcal{B}_{H,b}} |E(T)|.$$

By analyzing restrictive partial colorings on carefully chosen subtrees of trees T with $T \rightsquigarrow (H; b)$, I have been able to verify this conjecture for a few interesting classes of trees, including trees of diameter at most 4.

Theorem 1 (Picollelli [22]) *If H has diameter at most* 4*, then*

$$AR(H; b) = \min_{T \in \mathcal{B}_{H,b}} |E(T)|.$$

It would be interesting to determine whether the methods employed in [22] allow for further extension, in particular to trees of diameter 5, though I suspect this to be rather difficult.

Random Graphs with a Fixed Degree Sequence

Random graph models have received an enormous amount of study since the initial work of Erdős and Rényi (See [6] or [14], for example). In recent years, attention has turned to alternate models of sparse random graphs, such as graphs with a given degree sequence. Such models appear more suited to modeling large networks, where the properties of those networks differ significantly from those of dense random graphs (see, for example, [21]).

One model of particular interest to me is the $G_{\mathbf{z}}$ model, introduced in [2]: let $\mathbb{R}^+$ denote the nonnegative reals, Δ be a positive integer, and $\mathbf{z} \in (\mathbb{R}^+)^\Delta$ with $z_\Delta > 0$. The $G_{\mathbf{z}}$ model consists of graphs chosen uniformly at random from those with $\lfloor z_i n \rfloor$ or $\lceil z_i n \rceil$ vertices of degree i (the nearest-integer rounding is chosen to preserve parity), where n is an integer that tends to infinity. (A related model is considered in [8], in which the expected degree of a vertex is fixed, rather than the actual degree.)

As mentioned above, part of the focus on sparse random graphs is to more accurately model processes on large networks. To this end, consider the SIR model from mathematical epidemiology, applied to $G_{\mathbf{z}}$. This is a model for the spread of an infection on a network over time, where nodes in the network can be susceptible to infection (S), infected (I), or recovered (R). An infected node recovers completely (and never infects again) according to an exponential distribution with parameter ρ, and while infected it transmits infections to its neighbors independently according to an exponential distribution with parameter λ. A question which seems to have only recently been considered [24] asks for a characterization of the evolution of the infection on $G_{\mathbf{z}}$ over time. By applying techniques from branching process theory as well as the more recently developed differential equations method for random algorithm analysis [25], Tom Bohman and I have found that the evolution of the degree sequence over the life of the infection is highly concentrated around the solution of an associated system of ordinary differential equations [4].

While questions on connectivity [19] have been studied, there are several natural questions about the $G_{\mathbf{z}}$ model that remain open. A sequence of events $E(n), n = 1, 2, \ldots$, is said to occur *with high probability* (w.h.p.) if $\Pr(E(n)) \to 1$ as $n \to \infty$.

Problem 1 *For which $\mathbf{z}$ does $G_{\mathbf{z}}$ have a perfect matching w.h.p.?*

Bohman and Frieze [2] analyzed the performance of the Karp-Sipser algorithm on $G_{\mathbf{z}}$ for log-concave distributions $\mathbf{z}$. They conjectured that $G_{\mathbf{z}}$ has a perfect matching when $z_1 = z_2 = 0$, i.e. $G_{\mathbf{z}}$ has minimum degree 3, but I have produced counterexamples for every minimum degree $\delta \geq 3$ for which $G_{\mathbf{z}}$ does not have even an almost-perfect matching, i.e. a matching of size $(1 - o(1))n/2$.

The question of whether perfect matchings exist invariably leads to that of whether Hamilton cycles exist.

Problem 2 *For which $\mathbf{z}$ is $G_{\mathbf{z}}$ Hamiltonian w.h.p.?*

When $0 = z_1 = z_2 = \cdots = z_{\Delta-1}$, $z_\Delta = 1$, $G_{\mathbf{z}}$ is the uniform model for random Δ-regular graphs, for which significantly more is known (see [26] for a recent survey). In particular, Wormald and Robinson [27] showed that almost all random regular graphs are Hamiltonian. Furthering that, Wormald [26] conjectures that random graphs with half of the vertices of degree three and half of degree four are Hamiltonian.

Conjecture 2 (Wormald [26]) *$G_{(0,0,1,1)}$ is Hamiltonian w.h.p..*

Other natural questions that arise are on the chromatic number and independence number of $G_{\mathbf{z}}$.

Problem 3 *For a given $\mathbf{z}$, what is the chromatic number of $G_{\mathbf{z}}$?*

Problem 4 *For a given $\mathbf{z}$, what is the independence number of $G_{\mathbf{z}}$?*

In the case of random regular graphs, Problem 3 has recently been resolved (see [23] and [1]). Problem 4 remains open even for $z = (0, 0, 1)$, i.e. random 3-regular graphs, with the best current bounds of $.4328 \leq \alpha(G_{(0,0,1)})/n \leq .4554$ w.h.p. (see [12] for the lower bound and [18] for the upper bound). Analogous results for the $G_{\mathbf{z}}$ model, however, have not yet been found.

Finally, the differential equations method has become an invaluable tool for studying $G_{\mathbf{z}}$ (see [2], [23], or [19], for example), and I am interested in further applications of it, both to these and other problems. Additionally, it would be interesting to extend the known results on $G_{\mathbf{z}}$, such as the analysis of Karp-Sipser from [2], to models with arbitrarily large degrees, such as those with distributions $\mathbf{z} \in (\mathbb{R}^+)^\infty$ for which $\sum_{i=1}^{\infty} z_i < \infty$.

Extremal Set Theory

Extremal set theory, on the whole, is the study of the maximum (or minimum) size of families of sets satisfying certain conditions, most often on the intersection of its members. Perhaps the most famous and fundamental result in extremal set theory is the Erdős-Ko-Rado Theorem (See 8.1 in [15]):

Theorem 2 (Erdős-Ko-Rado) *If n and k are positive integers with $n \geq 2k$, and $\mathcal{F}$ is a family of k-element subsets of an n-element set with $A \cap B \neq \emptyset$ for all $A, B \in \mathcal{F}$, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$. Furthermore, if $n > 2k$ and $|\mathcal{F}| = \binom{n-1}{k-1}$, then $\mathcal{F}$ consists of all k-element subsets containing a given element.*

Various generalizations of the Erdős-Ko-Rado Theorem have been proposed, including the following, by Chvátal [7]. Define a d-simplex to be a collection of $d + 1$ distinct sets, $A_1, \ldots, A_{d+1}$, such that any d of them have nonempty intersection, but $\bigcap_{i=1}^{d+1} A_i = \emptyset$. Additionally, let $[n] = \{1, 2, \ldots, n\}$. Let $f(n, k, d)$ be the maximum size of a family $\mathcal{F}$ of k-element subsets of $[n]$ containing no d-simplex. Thus, by Theorem 2, if $n \geq 2k$, $f(n, k, 1) = \binom{n-1}{k-1}$.

Erdős [9] conjectured that $f(n, k, 2) = \binom{n-1}{k-1}$, which was recently settled by Mubayi and Verstraëte [20]. Chvátal, in attacking Erdős's conjecture, proved the following:

Theorem 3 (Chvátal [7]) *If $n \geq k + 2 \geq 5$, then $f(n, k, k - 1) = \binom{n-1}{k-1}$.*

In the same paper, he posed the following conjecture.

Conjecture 3 (Chvátal [7]) *If $k \geq d + 1 \geq 2$ and $n > (\frac{d+1}{d})k$, then $f(n, k, d) = \binom{n-1}{k-1}$. Moreover, if $\mathcal{F}$ is a family of k-element subsets of $[n]$ containing no d-simplex and $|\mathcal{F}| = \binom{n-1}{k-1}$, then $\mathcal{F} = \{A \subset [n] : |A| = k, a \in A\}$ for some $a \in [n]$.*

Progress on Conjecture 3 was made by Frankl and Füredi [11], who showed it when n is a sufficiently large function of k, and by Keevash and Mubayi [16], who showed it for sufficiently large n in the case that k and $n/2 - k$ are both bounded away from 0. However, Conjecture 3 is still open for small values of n and k (with $d \geq 3$, of course), such as $n = 7, k = 5, d = 3$.

The function $f(n, k, d)$ is itself closely linked with another active area of research, hypergraph Turán theory. In particular, $f(n, k, k)$ is the maximum size of a k-uniform hypergraph containing no complete sub-hypergraph on $k + 1$ vertices. It is known that $f(n, 2, 2) = \lfloor \frac{n^2}{4} \rfloor$, an immediate consequence of Turán's Theorem for graphs (originally proved by Mantel, see Chapter VI of [5]). On the other hand, determining $f(n, k, k)$ remains open for all $k \geq 3$.

Another problem of interest to me, related to the above question, was posed by Erdős [9]: let $g(n)$ denote the minimum integer m such that for any family $\mathcal{F}$ of subsets of $[n]$ with $|\mathcal{F}| = m$, there are three elements $a_1, a_2, a_3 \in [n]$ and three sets $A_1, A_2, A_3 \in \mathcal{F}$ such that $A_i \cap \{a_1, a_2, a_3\} = \{a_1, a_2, a_3\} \setminus \{a_i\}$.

Problem 5 (Erdős [9]) *Determine $g(n)$ for all n.*

The problem of even producing asymptotically sharp bounds for $g(n)$ appears to be completely untreated in the literature. Consider now a further restriction, letting $g(n, k)$ denote the same function with the addition that $\mathcal{F}$ consists only of k-element subsets of $[n]$. Again, by Turán's Theorem, one can easily see that $g(n, 2) = \lfloor \frac{n^2}{4} \rfloor + 1$. When $k = 3$, Theorem 3 implies that for $n \geq 5$, $g(n, 3) \leq \binom{n-1}{2} + 1$ for $n \geq 5$. However, Sauer [9] conjectures a different value for $g(n, 3)$, which I believe to be correct.

Conjecture 4 (Sauer [9]) $g(n, 3) = \lfloor \frac{(n-1)^2}{4} \rfloor + 1$.

References

[1] D. Achlioptas and C. Moore, *The Chromatic Number of Random Regular Graphs*, Proc. 8th RANDOM, 2004, pp. 219-228.

[2] T. Bohman and A. Frieze, *Karp-Sipser on random graphs with a fixed degree sequence*, manuscript.

[3] T. Bohman, A. Frieze, O. Pikhurko and C. Smyth, *Anti-Ramsey Properties of Random Graphs*, manuscript.

[4] T. Bohman and M. Picollelli, *The Spread of Disease on a Random Graph with a Fixed Degree Sequence,* in preparation.

[5] B. Bollobás. *Extremal Graph Theory,* New York: Academic Press, 1978.

[6] B. Bollobás, *Random Graphs*, Cambridge University Press (2001).

[7] V. Chvátal, *An extremal set-intersection theorem,* London Math. Soc, 9 (1974), 355-359.

[8] F. Chung, L. Lu, V. Vu, *The spectra of random graphs with given expected degrees,* Proceedings of National Academy of Sciences 100, no. 11, (2003), 6313-6318.

[9] P. Erdős, *Topics in combinatorial analysis*, Proc. of The Second Louisiana Conference on Combinatorics, Graph Theory and Computing (Baton Rouge, 1971), 2-20.

[10] P. Erdős, M. Simonovits and V. T. Sós, *Anti-Ramsey Theorems*, Colloquia Mathematica Societatis János Bolya 10, Infinite and Finite Sets, Keszethely, 1973.

[11] P. Frankl and Z. Füredi, *Exact solution of some Turán-type problems*, J. Combin. Theory Ser. A. 45 (1987), 226262.

[12] A. Frieze and S. Suen, *On the independence number of random cubic graphs*, Random Structures and Algorithms 5 (1994), 649-664.

[13] R. Graham, B. Rothschild and J. Spencer, *Ramsey Theory*, Wiley-Interscience Series in Discrete Mathematics, second ed. (1990), Wiley, New York, 1980.

[14] S. Janson, T. Łuczak and A. Ruciński, *Random Graphs*. John Wiley and Sons, 2000.

[15] S. Jukna, *Extremal Combinatorics With Applications in Computer Science*, Springe-Verlag, 2001.

[16] P. Keevash, D. Mubayi, *Set systems without a simplex or a cluster*, manuscript.

[17] H. Lefmann, V. Rödl and B. Wysocka, *Multicolored Subsets in Colored Hypergraphs*, Journal of Combinatorial Theory A 74 (1996) 209-248.

[18] B. D. McKay, *Independent sets in regular graphs of high girth*, in Proc. Australia-Singapore Joint Conference on Information Processing and Combinatorial Mathematics (Singapore, 1986). Ars Combinatoria, 23A, (1987), pp. 179-185.

[19] M. Molloy and B. Reed, *A Critical Point for Random Graphs with a Given Degree Sequence*, Random Structures and Algorithms 6 (1996) 161-180.

[20] D. Mubayi and J. Verstraëte, *Proof of a conjecture of Erdős on triangles in set-systems*, Combinatorica 25 (2005), 599614.

[21] M. Newman, *Random graphs as models of networks*, in Handbook of Graphs and Networks (ed. S. Bornholdt and H. G. Schuster), Weinheim: Wiley, pp. 35-65.

[22] M. Picollelli, *An Anti-Ramsey Condition on Trees*, submitted.

[23] L. Shi and N. Wormald, *Colouring random regular graphs,* manuscript.

[24] Volz, E. *SIR Dynamics in Random Networks with Heterogeneous Connectivity,* Journal of Mathematical Biology, DOI: 10.1007/s00285-007-0116-4 (Published online: 1 August 2007)

[25] N.C. Wormald, *The Differential Equation Method for Random Graph Processes and Greedy Algorithms*, in Lectures on Approximation and Randomized Algorithms, M. Karonski and H.J. Promel, editors, 1999, pp. 73-155.

[26] N.C. Wormald, *Models of random regular graphs*, in Surveys in Combinatorics, 1999, J.D. Lamb and D.A. Preece, eds. London Mathematical Society Lecture Note Series, vol 276, pp. 239-298. Cambridge University Press, Cambridge, 1999.

[27] R.W. Robinson and N.C. Wormald, *Almost all regular graphs are hamiltonian,* Random Structures and Algorithms 5 (1994), 363-374.