

# SUPERSTABILITY, NOETHERIAN RINGS AND PURE-SEMISIMPLE RINGS

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ABSTRACT. We uncover a connection between the model-theoretic notion of superstability and that of noetherian rings and pure-semisimple rings.

We obtain a characterization of noetherian rings via superstability of the class of left modules with embeddings.

**Theorem 0.1.** *For a ring  $R$  the following are equivalent.*

- (1)  $R$  is left noetherian.
- (2) The class of left  $R$ -modules with embeddings is superstable.
- (3) For every  $\lambda \geq |R| + \aleph_0$ , there is a  $\chi \geq \lambda$  such that the class of left  $R$ -modules with embeddings has uniqueness of limit models of cardinality  $\chi$ .

We obtain a characterization of left pure-semisimple rings via superstability of the class of left modules with pure embeddings.

**Theorem 0.2.** *For a ring  $R$  the following are equivalent.*

- (1)  $R$  is left pure-semisimple.
- (2) The class of left  $R$ -modules with pure embeddings is superstable.
- (3) There exists  $\lambda \geq (|R| + \aleph_0)^+$  such that the class of left  $R$ -modules with pure embeddings has uniqueness of limit models of cardinality  $\lambda$ .

We think that both equivalences provide evidence that that the notion of superstability could shed light in the understanding of algebraic concepts.

As this paper is aimed at model theorists and algebraists an effort was made to provide the background for both.

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## 1. INTRODUCTION

An abstract elementary class (AEC) is a pair  $\mathbf{K} = (K, \leq_{\mathbf{K}})$ , where  $K$  is class of structures and  $\leq_{\mathbf{K}}$  is a strong substructure relation extending the substructure relation. Among the requirements we have that an AEC is closed under colimits and that every set is contained in a

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small model in the class (see Definition 2.1). These were introduced by Shelah in [Sh87a]. Natural examples in the context of algebra are abelian groups with embeddings, torsion-free abelian groups with pure embeddings, first-order axiomatizable classes of modules with pure embeddings and flat modules with flat embeddings.<sup>2</sup>

*Limit models* were introduced by Kolman and Shelah in [KolSh96] as a substitute for saturated models in the context of abstract elementary classes. Intuitively the reader can think of them as universal models with some level of homogeneity (see Definition 2.8). Limit models in classes of abelian groups with embeddings are divisible groups [Maz, 3.5] and *long* limit models in classes of modules with pure embeddings are pure-injective modules [KuMa, 4.4]. The key question regarding limit models is the uniqueness of limit models of a given cardinality but with chains of different lengths. This has been studied thoroughly in the context of abstract elementary classes [ShVi99], [Van06], [GVV16], [Bon14a], [Van16], [BoVan] and [Vas].

Dividing lines in complete first-order theories were introduced by Shelah in [Sh78]. One of the best behaved classes is that of *superstable* theories. Extensions of superstability in a non-elementary setting were first considered in [GrSh86]. In the context of AECs, superstability was introduced in [Sh99] and until recently it was believed to suffer from “schizophrenia” [Sh:h, p. 19]. In [GrVas17, 1.3] and [Vas18], it was shown (under additional hypothesis that are satisfied by the classes studied in this paper<sup>3</sup>) that superstability is a well-behaved concept and many conditions that were believed to characterize superstability were found to be equivalent. Based on this and the key role limit models play in this paper, we will say that an AEC is *superstable* if it has uniqueness of limit models in a tail of cardinals.<sup>4</sup> Further details on the development of the notion of superstability can be consulted in the introduction of [GrVas17].

In this paper, we show that the notion of superstability has algebraic substance if one chooses the right context. More specifically, we characterize noetherian rings and pure-semisimple rings via superstability in the class of modules with embeddings and with pure embeddings respectively.

*Noetherian rings* are rings with the ascending chain condition for ideals. The precise equivalence we obtain is the following<sup>5</sup>.

**Theorem 3.12.** *For a ring  $R$  the following are equivalent.*

- (1)  *$R$  is left noetherian.*
- (2) *The class of left  $R$ -modules with embeddings is superstable.*
- (3) *For every  $\lambda \geq |R| + \aleph_0$ , there is a  $\chi \geq \lambda$  such that the class of left  $R$ -modules with embeddings has uniqueness of limit models of cardinality  $\chi$ .*
- (4) *For every  $\lambda \geq |R| + \aleph_0$ , the class of left  $R$ -modules with embeddings has uniqueness of limit models of cardinality  $\lambda$ .*
- (5) *For every  $\lambda \geq (|R| + \aleph_0)^+$ , the class of left  $R$ -modules with embeddings has a superlimit of cardinality  $\lambda$ .*
- (6) *For every  $\lambda \geq |R| + \aleph_0$ , the class of left  $R$ -modules with embeddings is stable.*

A ring  $R$  is *left pure-semisimple* if every left  $R$ -module is pure-injective. It was pointed out to us by Daniel Simson that pure-semisimple rings were introduced by him in [Sim77]. There are many papers where equivalent conditions for pure-semisimple rings are studied, for example [Cha60], [Aus74], [Aus76], [Zim79], [Sim81] and [Pre84]. For additional information on what is known about pure-semisimple rings the reader can consult [Sim00], [Hui00] and [Pre09, §4.5.1].

<sup>2</sup>The first two classes were studied in [BCG+], [BET07] and [Maz], the next one was studied in [KuMa] and the last one was studied in [LRV, §6].

<sup>3</sup>The hypothesis are amalgamation, joint embedding, no maximal models and  $\text{LS}(\mathbf{K})$ -tameness.

<sup>4</sup>For a complete first-order theory  $T$ ,  $(\text{Mod}(T), \preceq)$  is superstable if and only if  $T$  is superstable as a first-order theory, i.e.,  $T$  is  $\lambda$ -stable as for every  $\lambda \geq 2^{|T|}$ .

<sup>5</sup>Conditions (4) through (6) of the theorem below were motivated by [GrVas17, 1.3].

In this paper, we give a few new characterizations of left pure-semisimple rings via superstability. More precisely, we show<sup>6</sup>.

**Theorem 4.25.** *For a ring  $R$  the following are equivalent.*

- (1)  $R$  is left pure-semisimple.
- (2) The class of left  $R$ -modules with pure embeddings is superstable.
- (3) There exists  $\lambda \geq (|R| + \aleph_0)^+$  such that the class of left  $R$ -modules with pure embeddings has uniqueness of limit models of cardinality  $\lambda$ .
- (4) For every  $\lambda \geq |R| + \aleph_0$ , the class of left  $R$ -modules with pure embeddings has uniqueness of limit models of cardinality  $\lambda$ .
- (5) For every  $\lambda \geq (|R| + \aleph_0)^+$ , the class of left  $R$ -modules with pure embeddings has a superlimit of cardinality  $\lambda$ .
- (6) For every  $\lambda \geq |R| + \aleph_0$ , the class of left  $R$ -modules with pure embeddings is  $\lambda$ -stable.
- (7) For every  $\lambda \geq (|R| + \aleph_0)^+$ , an increasing chain of  $\lambda$ -saturated models is  $\lambda$ -saturated in the class of left  $R$ -modules with pure embeddings.

A key difference between our results and those of [GrVas17, 1.3] is that in [GrVas17] the cardinal where the *nice property* starts to show up is eventual (bounded by  $\beth_{(2^{|R| + \aleph_0})^+}$ ), while in our case the cardinal is exactly  $|R| + \aleph_0$  or  $(|R| + \aleph_0)^+$ . In the introduction of [GrVas17] is asked if it was possible to lower these bounds (see Theorem 4.22 and the remark below it).

Although the results of Theorem 3.12 and Theorem 4.25 are similar, the techniques used to prove the results differ significantly. The proof of Theorem 3.12 is more rudimentary and depends heavily on the fact that we are working with the class of all modules. On the other hand, Theorem 4.25 is a corollary of the theory of superstable classes of modules with pure embeddings closed under direct sums which is developed in the fourth section of the paper. One could give a proof of Theorem 4.25 similar to that of Theorem 3.12, but we believe that the theory of superstable classes of modules with pure embeddings is an interesting theory that should be developed.

Another result of the paper is a positive solution above  $\text{LS}(\mathbf{K})^+$  to Conjecture 2 of [BoVan] in the case of classes of modules axiomatizable in first-order with joint embedding and amalgamation (see Theorem 4.28). This also provides a partial solution to Question 4.11 of [KuMa].

The paper is divided into four sections. Section 2 presents necessary background. Section 3 provides a new characterization of noetherian rings. Section 4 characterizes superstability with pure embeddings in classes of modules closed under direct sums and provides a new characterization of pure-semisimple rings. Moreover, a positive solution above  $\text{LS}(\mathbf{K})^+$  to Conjecture 2 of [BoVan] is given for certain classes of modules.

It was pointed out to us by Vasey that already in [Sh75] Shelah has a theorem on page 299 (Theorem 8.6) related to our Theorem 3.12. Immediately after that unproven theorem he has a remark (also on page 299) that indicates that he knew that superstability of the theory of modules implies that the ring is left noetherian. The precise equivalence he noticed is similar to that of (6) implies (1) of Theorem 3.12. It is clear that Shelah did not know that the two notions are equivalent.

This paper was written while the author was working on a Ph.D. under the direction of Rami Grossberg at Carnegie Mellon University and I would like to thank Professor Grossberg for his guidance and assistance in my research in general and in this work in particular. After reading a preliminary preprint, Sebastien Vasey informed us that he independently discovered the equivalence between superstability and noetherian rings (the equivalence between (1) and (2) of Theorem 3.12), but has not circulated it yet. His proof follows from [Vas17b, 3.7], [Ekl71, Proposition 3] and [Vas18, 5.9]. I thank an anonymous referee for comments on another paper of mine that prompted the development of Subsection 4.4. I would also like to thank John T. Baldwin,

<sup>6</sup>Conditions (4) through (7) of the theorem below were motivated by [GrVas17, 1.3].

Daniel Simson and Sebastien Vasey for comments that helped improve the paper. Finally, I would like to dedicate this work to Marquitos, you will always be loved and remembered.

## 2. PRELIMINARIES

We present the basic concepts of abstract elementary classes that are used in this paper. These are further studied in [Bal09, §4 - 8] and [Gro1X, §2, §4.4]. Regarding the background on module theory, we give a brief survey of the concepts we will use in this paper. An excellent resource for the module theory we will use in this paper are [Pre88] and [Pre09].

**2.1. Basic concepts.** Abstract elementary classes (AECs) were introduced by Shelah in [Sh87a, 1.2]. Among the requirements we have that an AEC is closed under colimits and that every set is contained in a small model in the class. Given a model  $M$ , we will write  $|M|$  for its underlying set and  $\|M\|$  for its cardinality.

**Definition 2.1.** *An abstract elementary class is a pair  $\mathbf{K} = (K, \leq_{\mathbf{K}})$ , where:*

- (1)  $K$  is a class of  $\tau$ -structures, for some fixed language  $\tau = \tau(\mathbf{K})$ .
- (2)  $\leq_{\mathbf{K}}$  is a partial order on  $K$ .
- (3)  $(K, \leq_{\mathbf{K}})$  respects isomorphisms:  
*If  $M \leq_{\mathbf{K}} N$  are in  $K$  and  $f : N \cong N'$ , then  $f[M] \leq_{\mathbf{K}} N'$ .  
 In particular (taking  $M = N$ ),  $K$  is closed under isomorphisms.*
- (4) *If  $M \leq_{\mathbf{K}} N$ , then  $M \subseteq N$ .*
- (5) *Coherence: If  $M_0, M_1, M_2 \in K$  satisfy  $M_0 \leq_{\mathbf{K}} M_2$ ,  $M_1 \leq_{\mathbf{K}} M_2$ , and  $M_0 \subseteq M_1$ , then  $M_0 \leq_{\mathbf{K}} M_1$ .*
- (6) *Tarski-Vaught axioms: Suppose  $\delta$  is a limit ordinal and  $\{M_i \in K : i < \delta\}$  is an increasing chain. Then:*
  - (a)  $M_\delta := \bigcup_{i < \delta} M_i \in K$  and  $M_i \leq_{\mathbf{K}} M_\delta$  for every  $i < \delta$ .
  - (b) *Smoothness: If there is some  $N \in K$  so that for all  $i < \delta$  we have  $M_i \leq_{\mathbf{K}} N$ , then we also have  $M_\delta \leq_{\mathbf{K}} N$ .*
- (7) *Löwenheim-Skolem-Tarski axiom: There exists a cardinal  $\lambda \geq |\tau(\mathbf{K})| + \aleph_0$  such that for any  $M \in K$  and  $A \subseteq |M|$ , there is some  $M_0 \leq_{\mathbf{K}} M$  such that  $A \subseteq |M_0|$  and  $\|M_0\| \leq |A| + \lambda$ . We write  $\text{LS}(\mathbf{K})$  for the minimal such cardinal.*

**Notation 2.2.**

- *If  $\lambda$  is a cardinal and  $\mathbf{K}$  is an AEC, then  $\mathbf{K}_\lambda = \{M \in \mathbf{K} : \|M\| = \lambda\}$ .*
- *Let  $M, N \in \mathbf{K}$ . If we write “ $f : M \rightarrow N$ ” we assume that  $f$  is a  $\mathbf{K}$ -embedding, i.e.,  $f : M \cong f[M]$  and  $f[M] \leq_{\mathbf{K}} N$ . In particular  $\mathbf{K}$ -embeddings are always monomorphisms.*

In [Sh87b] Shelah introduced a notion of semantic type. The original definition was refined and extended by many authors who following [Gro02] call these semantic types Galois-types (Shelah recently named them orbital types). We present here the modern definition and call them Galois-types throughout the text. We follow the notation of [MaVa18, 2.5].

**Definition 2.3.** *Let  $\mathbf{K}$  be an AEC.*

- (1) *Let  $\mathbf{K}^3$  be the set of triples of the form  $(\mathbf{b}, A, N)$ , where  $N \in \mathbf{K}$ ,  $A \subseteq |N|$ , and  $\mathbf{b}$  is a sequence of elements from  $N$ .*
- (2) *For  $(\mathbf{b}_1, A_1, N_1), (\mathbf{b}_2, A_2, N_2) \in \mathbf{K}^3$ , we say  $(\mathbf{b}_1, A_1, N_1)E_{at}(\mathbf{b}_2, A_2, N_2)$  if  $A := A_1 = A_2$ , and there exists  $f_\ell : N_\ell \xrightarrow[A]{} N$  such that  $f_1(\mathbf{b}_1) = f_2(\mathbf{b}_2)$ .*
- (3) *Note that  $E_{at}$  is a symmetric and reflexive relation on  $\mathbf{K}^3$ . We let  $E$  be the transitive closure of  $E_{at}$ .*
- (4) *For  $(\mathbf{b}, A, N) \in \mathbf{K}^3$ , let  $\text{gtp}_{\mathbf{K}}(\mathbf{b}/A; N) := [(\mathbf{b}, A, N)]_E$ . We call such an equivalence class a Galois-type. Usually,  $\mathbf{K}$  will be clear from context and we will omit it.*

- (5) For  $M \in \mathbf{K}$ ,  $\mathbf{gS}_{\mathbf{K}}(M) = \{\mathbf{gtp}_{\mathbf{K}}(b/M; N) : M \leq_{\mathbf{K}} N \in \mathbf{K} \text{ and } b \in N\}$ .  
(6) For  $\mathbf{gtp}_{\mathbf{K}}(\mathbf{b}/A; N)$  and  $C \subseteq A$ ,  $\mathbf{gtp}_{\mathbf{K}}(\mathbf{b}/A; N) \upharpoonright_C := [(\mathbf{b}, C, N)]_E$ .

**Definition 2.4.** An AEC is  $\lambda$ -stable if for any  $M \in \mathbf{K}_{\lambda}$ ,  $|\mathbf{gS}_{\mathbf{K}}(M)| \leq \lambda$ .

**Remark 2.5.** Recall that given  $T$  a complete first-order theory and  $A \subseteq M$  with  $M$  a model of  $T$ ,  $S^T(A)$  is the set of complete first-order types with parameters in  $A$ . For a complete first-order theory  $T$  and  $\lambda \geq |T|$ ,  $(\text{Mod}(T), \preceq)$  is  $\lambda$ -stable (where  $\preceq$  is the elementary substructure relation) if and only if  $T$  is  $\lambda$ -stable as a first-order theory, i.e.,  $|S^T(A)| \leq \lambda$  for every  $A \subseteq M$  with  $|A| = \lambda$  and  $M$  a model of  $T$ .

The following notion was isolated by Grossberg and VanDieren in [GrVan06].

**Definition 2.6.**  $\mathbf{K}$  is  $(< \kappa)$ -tame if for any  $M \in \mathbf{K}$  and  $p \neq q \in \mathbf{gS}(M)$ , there is  $A \subseteq |M|$  such that  $|A| < \kappa$  and  $p \upharpoonright_A \neq q \upharpoonright_A$ .

**2.2. Limit models, saturated models and superlimits.** Before introducing the concept of limit model we recall the concept of universal extension.

**Definition 2.7.**  $M$  is universal over  $N$  if and only if  $N \leq_{\mathbf{K}} M$ ,  $\|M\| = \|N\| = \lambda$  and for any  $N^* \in \mathbf{K}_{\lambda}$  such that  $N \leq_{\mathbf{K}} N^*$ , there is  $f : N^* \xrightarrow[N]{} M$ .

With this we are ready to introduce limit models, they were originally introduced in [KolSh96].

**Definition 2.8.** Let  $\alpha < \lambda^+$  a limit ordinal.  $M$  is a  $(\lambda, \alpha)$ -limit model over  $N$  if and only if there is  $\{M_i : i < \alpha\} \subseteq \mathbf{K}_{\lambda}$  an increasing continuous chain such that  $M_0 := N$ ,  $M_{i+1}$  is universal over  $M_i$  for each  $i < \alpha$  and  $M = \bigcup_{i < \alpha} M_i$ . We say that  $M \in \mathbf{K}_{\lambda}$  is a  $(\lambda, \alpha)$ -limit model if there is  $N \in \mathbf{K}_{\lambda}$  such that  $M$  is a  $(\lambda, \alpha)$ -limit model over  $N$ . We say that  $M \in \mathbf{K}_{\lambda}$  is a limit model if there is  $\alpha < \lambda^+$  limit ordinal such that  $M$  is a  $(\lambda, \alpha)$ -limit model.

**Definition 2.9.** Let  $\mathbf{K}$  an AEC and  $\lambda$  a cardinal.  $M \in \mathbf{K}$  is a universal model in  $\mathbf{K}_{\lambda}$  if  $M \in \mathbf{K}_{\lambda}$  and if given any  $N \in \mathbf{K}_{\lambda}$ , there is  $f : N \rightarrow M$ .

The following is a simple exercise, a proof is given in [Maz, 2.10].

**Fact 2.10.** Let  $\mathbf{K}$  be an AEC with joint embedding and amalgamation. If  $M$  is a limit model of cardinality  $\lambda$ , then  $M$  is a universal model in  $\mathbf{K}_{\lambda}$ .

The next fact gives conditions for the existence of limit models.

**Fact 2.11** ([Sh:h, §II], [GrVan06, 2.9]). Let  $\mathbf{K}$  be an AEC with joint embedding, amalgamation and no maximal models. If  $\mathbf{K}$  is  $\lambda$ -stable, then for every  $N \in \mathbf{K}_{\lambda}$  and  $\alpha < \lambda^+$  limit ordinal there is  $M$  a  $(\lambda, \alpha)$ -limit model over  $N$ . Conversely, if  $\mathbf{K}$  has a limit model of cardinality  $\lambda$ , then  $\mathbf{K}$  is  $\lambda$ -stable

As mentioned in the introduction, the key question is the uniqueness of limit models of the same cardinality but with chains of different lengths.

**Definition 2.12.**  $\mathbf{K}$  has uniqueness of limit models of cardinality  $\lambda$  if  $\mathbf{K}$  has a limit model of cardinality  $\lambda$  and if given  $M, N \in \mathbf{K}_{\lambda}$  limit models,  $M$  and  $N$  are isomorphic.

Since we will only deal with AECs with amalgamation, joint embedding, no maximal models and  $\text{LS}(\mathbf{K})$ -tame and it is known (by [GrVas17, 1.3] and [Vas18]) that in this context the definition below is equivalent to every other definition of superstability considered in the context of AECs, we introduce the following as the definition of superstability.

**Definition 2.13.**  $\mathbf{K}$  is superstable if and only if  $\mathbf{K}$  has uniqueness of limit models in a tail of cardinals.

**Remark 2.14.** For a complete first-order  $T$ ,  $(\text{Mod}(T), \preceq)$  is superstable if and only if  $T$  is superstable as a first-order theory, i.e.,  $T$  is  $\lambda$ -stable for every  $\lambda \geq 2^{|T|}$ .

It is important to point out that to establish that  $\mathbf{K}$  has the property of uniqueness of limit models of cardinality  $\lambda$ , one needs to show first the existence of limit models. Due to Fact 2.11, this is equivalent to  $\lambda$ -stability.

Another important class of models is that of saturated models.

**Definition 2.15.**  $M \in \mathbf{K}$  is  $\lambda$ -saturated if for every  $N \leq_{\mathbf{K}} M$  and  $p \in \mathbf{gS}(N)$  with  $\|N\| < \lambda$ , there is  $a \in M$  such that  $p = \mathbf{gtp}(a/N; M)$ .  $M$  is saturated if  $M$  is  $\|M\|$ -saturated.

A model  $M$  is  $\lambda$ -model-homogeneous if for every  $N, N' \in \mathbf{K}$  with  $N \leq_{\mathbf{K}} M$ ,  $N \leq_{\mathbf{K}} N'$  and  $\|N'\| < \lambda$ , there is  $f : N' \xrightarrow[N]{\cong} M$ . Recall that for  $\lambda > \text{LS}(\mathbf{K})$ , a model is  $\lambda$ -saturated if and only if it is  $\lambda$ -model-homogeneous. A proof of it appears in [Sh:h, §II.1.4].

Superlimit models were introduced in [Sh87a, 3.1.(1)] as another possible notion of saturation on AECs.

**Definition 2.16.** Let  $\mathbf{K}$  an AEC. Let  $M \in \mathbf{K}$  and  $\lambda \geq \text{LS}(\mathbf{K})$ .  $M$  is superlimit in  $\lambda$  if:

- (1)  $M \in \mathbf{K}_\lambda$ .
- (2) For every  $N \in \mathbf{K}_\lambda$ , there is  $f : N \rightarrow M$  such that  $f[N] \neq M$ .
- (3) If  $\{M_i : i < \delta\} \subseteq \mathbf{K}_\lambda$  is an increasing chain,  $\delta < \lambda^+$  and  $M_i \cong M$  for all  $i < \delta$ , then  $\bigcup_{i < \delta} M_i \cong M$ .

The following fact has some known connections between limit models, saturated models and superlimits.

**Fact 2.17** ([GrVas17, 2.8], [Dru13, 2.3.10]). Let  $\mathbf{K}$  an AEC with amalgamation, joint embedding and no maximal models.

- (1) If  $\lambda > \text{LS}(\mathbf{K})$  and  $M$  is a  $(\lambda, \alpha)$ -limit model for  $\alpha \in [\text{LS}(\mathbf{K})^+, \lambda]$  a regular cardinal, then  $M$  is an  $\alpha$ -saturated model.
- (2) Let  $\lambda > \text{LS}(\mathbf{K})$  and  $\mathbf{K}$  be  $\lambda$ -stable.  $\mathbf{K}$  has uniqueness of limit models in  $\lambda$  if and only if every limit model of cardinality  $\lambda$  is saturated.
- (3) Let  $\mathbf{K}$  be  $\lambda$ -stable. If  $M$  is a superlimit of cardinality  $\lambda$ , then  $M$  is a  $(\lambda, \alpha)$ -limit model for every  $\alpha < \lambda^+$  limit ordinal.
- (4) Let  $\lambda > \text{LS}(\mathbf{K})$ ,  $\mathbf{K}$  be  $\lambda$ -stable and there exists a saturated model of size  $\lambda$ .  $\mathbf{K}$  has a superlimit of cardinality  $\lambda$  if and only if the union of an increasing chain (of length less than  $\lambda^+$ ) of saturated models in  $\mathbf{K}_\lambda$  is saturated.

**2.3. Module theory.** All rings considered in this paper are associative with an identity element. A ring  $R$  is *left noetherian* if every increasing chain of left ideals is stationary. This were introduced by Noether in [Noe21].

A module  $M$  is *injective* if and only if for every module  $N$ , if  $M \leq N$  then  $M$  is a direct summand of  $N$ . We say that  $M$  is  $\Sigma$ -*injective* if and only if  $M^{(I)}$  is injective for every index set  $I$ . To consider only countable index sets one needs  $M$  to be injective.

**Fact 2.18** ([Fai66, Proposition 3]). For  $M$  injective the following are equivalent.

- (1)  $M$  is  $\Sigma$ -injective.
- (2)  $M^{(\aleph_0)}$  is injective.

Following [Ekl71], denote by  $\gamma_R$  the smallest cardinal such that every left ideal of  $R$  is generated by less than  $\gamma_R$  elements. Observe that  $\gamma_R \leq |R|^+$ . We will use the following equivalence in Section 3. The equivalence between one and four is due to Cartan-Eilenberg-Bass-Papp and the equivalence between one and two is trivial

**Fact 2.19** ([FaWa67]). *For a ring  $R$  the following are equivalent.*

- (1)  $R$  is left noetherian.
- (2)  $\gamma_R \leq \aleph_0$ .
- (3) Every injective left  $R$ -module is  $\Sigma$ -injective.
- (4) Every direct sum of injective left  $R$ -modules is injective.

Recall that a formula  $\phi$  is a positive primitive formula (*pp*-formula for short), if  $\phi$  is an existentially quantified system of linear equations. Given  $M$  and  $N$   $R$ -modules,  $M$  is a *pure submodule* of  $N$ , denoted by  $M \leq_{pp} N$ , if and only if  $M$  is a submodule of  $N$  and for every *pp*-formula  $\phi$  it holds that  $\phi[N] \cap M = \phi[M]$ . Equivalently if for every  $L$  right  $R$ -module  $L \otimes M \rightarrow L \otimes N$  is a monomorphism.

( $\Sigma$ -)Pure-injective modules generalize the notion of ( $\Sigma$ -)injective modules. A module  $M$  is *pure-injective* if in the definition of injective module one substitutes “ $\leq$ ” by “ $\leq_{pp}$ ”. A module  $M$  is  *$\Sigma$ -pure-injective* if in the definition of  $\Sigma$ -injective module one substitutes “injective” for “pure-injective”. In the case of  $\Sigma$ -pure-injectivity it is enough to consider countable index sets.

**Fact 2.20** ([Zim79, 3.4]).  *$M$  is  $\Sigma$ -pure-injective if and only if  $M^{(\aleph_0)}$  is pure-injective.*

Recall that a module  $M$  is *absolutely pure* if every extension of  $M$  is pure. The next fact relates  $\Sigma$ -injectivity and  $\Sigma$ -pure-injectivity.

**Fact 2.21** ([Pre09, 4.4.16]). *For  $M$  an  $R$ -module the following are equivalent.*

- (1)  $M$  is  $\Sigma$ -injective.
- (2)  $M$  is absolutely pure and  $\Sigma$ -pure-injective.

Using the equivalence between  $\Sigma$ -pure-injectivity and the descending chain condition on *pp*-definable subgroups one can show the following (see for example [Pre88, 2.11]).

**Fact 2.22.**

- If  $N$  is  $\Sigma$ -pure-injective and  $M \leq_{pp} N$ , then  $M$  is  $\Sigma$ -pure-injective.
- If  $N$  is  $\Sigma$ -pure-injective and  $M$  is elementary equivalent to  $N$ , then  $M$  is  $\Sigma$ -pure-injective.

We will also use that  $\Sigma$ -pure-injective modules are totally transcendental.

**Fact 2.23** ([Pre88, 3.2]). *If  $M$  is  $\Sigma$ -pure-injective, then  $(Th(M), \preceq)$  is  $\lambda$ -stable for every  $\lambda \geq |Th(M)|$ .*

Recall the notion of left pure-semisimple ring.

**Definition 2.24.** *A ring  $R$  is left pure-semisimple if and only if every left  $R$ -module  $M$  is pure-injective.*

Many equivalent conditions have been found for the notion of a pure-semisimple ring, see for example [Cha60], [Aus74], [Aus76], [Zim79], [Sim81] and [Pre84]. A more updated set of equivalences is given in [Sim00] and [Pre09, §4.5.1]. Below we give some of the equivalent conditions for a ring to be left pure-semisimple.

**Fact 2.25** ([Pre88, 11.3]). *For a ring  $R$  the following are equivalent.*

- (1)  $R$  is left pure-semisimple.
- (2) Every left  $R$ -module  $M$  is  $\Sigma$ -pure-injective.
- (3) Every left  $R$ -module is the direct sum of indecomposable submodules.

Recall the following generalization of Bumby’s result [Bum65] to pure-injective modules. A proof of it (and a discussion of the general setting) appears in [GKS, 3.2].

**Fact 2.26.** *Let  $M, N$  be pure-injective modules. If there are  $f : M \rightarrow N$  a pure embedding and  $g : N \rightarrow M$  a pure embedding, then  $M$  is isomorphic to  $N$ .*

**2.4. Notation.** We will use the following notation which was introduced in [KuMa, 3.1].

**Notation 2.27.** Given  $R$  a ring, we denote by  $\mathbf{Th}_R$  the theory of left  $R$ -modules. A (not necessarily complete) first-order theory  $T$  is a theory of modules if it extends  $\mathbf{Th}_R$ . For  $T$  a theory of modules, let  $\mathbf{K}^T = (\text{Mod}(T), \leq_{pp})$  and  $|T| = |R| + \aleph_0$ .

Since we will also work with embeddings we introduce the following notation.

**Notation 2.28.** Given  $R$  a ring, let  $\mathbf{K}^{\text{Th}_R, \leq} = (\text{Mod}(\mathbf{Th}_R), \leq)$ .

### 3. A NEW CHARACTERIZATION OF NOETHERIAN RINGS

In this section we will work in the class of modules with embeddings. Since complete theories of modules only have  $pp$ -quantifier elimination, we do not think that in the case of classes of modules with embeddings there is a deep theory as the one we will develop in the next section for pure embeddings. Instead, using some more rudimentary methods, we will study the class of modules with embeddings.

**Remark 3.1.** It is well-known that  $\mathbf{K}^{\text{Th}_R, \leq}$  is an AEC that has amalgamation, joint embedding and no maximal models.

The next assertion describes Galois-types in this context.

**Lemma 3.2.** Let  $M, N_1, N_2 \in \mathbf{K}^{\text{Th}_R, \leq}$ ,  $M \leq N_1, N_2$ ,  $\bar{b}_1 \in N_1^{<\omega}$  and  $\bar{b}_2 \in N_2^{<\omega}$ . Then:

$\text{gtp}_{\mathbf{K}^{\text{Th}_R, \leq}}(\bar{b}_1/M; N_1) = \text{gtp}_{\mathbf{K}^{\text{Th}_R, \leq}}(\bar{b}_2/M; N_2)$  if and only if  $qf\text{-tp}(\bar{b}_1/M, N_1) = qf\text{-tp}(\bar{b}_2/M, N_2)$ .

*Proof sketch.* The forward direction is trivial, so let us sketch the backward direction. By the amalgamation property we may assume that  $N_1 = N_2 = N$ . Define  $f : \langle \bar{b}_1 M \rangle \rightarrow \langle \bar{b}_2 M \rangle$  as  $f(\sum_{i=1}^n r_i b_{1,i} + \sum_{i=1}^k s_i m_i) = \sum_{i=1}^n r_i b_{2,i} + \sum_{i=1}^k s_i m_i$  where  $\langle \bar{b}_\ell M \rangle$  is the submodule generated by  $\bar{b}_\ell M$  inside  $N$  for  $\ell \in \{1, 2\}$ ,  $r_i, s_i \in R$  for all  $i$  and  $m_i \in M$  for all  $i$ . Using that the quantifier free types are equal, it follows that  $f$  is an isomorphism. Then the result follows by applying amalgamation a couple of times.  $\square$

Since we can witness that two Galois-types are different by a quantifier free formula, we obtain.

**Corollary 3.3.**  $\mathbf{K}^{\text{Th}_R, \leq}$  is  $(< \aleph_0)$ -tame.

The above corollary also follows from the general theory of AECs [Vas17a, 3.7], since  $\mathbf{K}^{\text{Th}_R, \leq}$  is a universal class in the sense of [Tar54] (see [MaVa18, 2.1] for the definition).

An analogous argument to the one given in [KuMa, 4.7] can be used to show the following.

**Proposition 3.4.** Let  $\lambda$  an infinite cardinal. If  $E \in \mathbf{K}_\lambda^{\text{Th}_R, \leq}$  is injective and  $U \in \mathbf{K}^{\text{Th}_R, \leq}$  is universal in  $\mathbf{K}_\lambda^{\text{Th}_R, \leq}$ , then  $E \oplus U$  is universal over  $E$ .

The next fact from [Ekl71] will be useful.

**Fact 3.5** ([Ekl71, Proposition 3]). Let  $\lambda$  an infinite cardinal with  $\lambda \geq |R| + \aleph_0$ .  $\lambda^{<\gamma_R} = \lambda$  if and only if there is an injective universal model of cardinality  $\lambda$ .

With it we will be able to show that  $\mathbf{K}^{\text{Th}_R, \leq}$  is stable.

**Lemma 3.6.** Let  $R$  be a ring and  $\lambda$  an infinite cardinal with  $\lambda \geq |R| + \aleph_0$ . If  $\lambda^{<\gamma_R} = \lambda$ , then  $\mathbf{K}^{\text{Th}_R, \leq}$  is  $\lambda$ -stable.

*Proof.* By Fact 3.5 there is  $U$  an injective universal model of cardinality  $\lambda$ . Build  $\{N_i : i < \omega\}$  by induction such that  $N_i$  is equal to  $i$ -many direct copies of  $U$ .

Since  $U$  is injective and injective objects are closed under finite direct sums, it follows that  $N_i$  is injective for every  $i < \omega$ . Moreover, by Proposition 3.4 it follows that  $N_{i+1}$  is universal over

$N_i$  for every  $i < \omega$ . Let  $N = \bigcup_{i < \omega} N_i$ . Observe that  $N$  is a  $(\lambda, \omega)$ -limit model, so by Fact 2.11 it follows that  $\mathbf{K}^{\text{Th}_{R, \leq}}$  is  $\lambda$ -stable.  $\square$

From the above theorem and Fact 2.11 it follows that there is a  $(\lambda, \alpha)$ -limit model for every  $\alpha < \lambda^+$  and  $\lambda$  such that  $\lambda^{<\gamma_R} = \lambda$ . The next lemma characterizes limit models in  $\mathbf{K}^{\text{Th}_{R, \leq}}$ .

**Lemma 3.7.** *Let  $R$  be a ring,  $\lambda$  an infinite cardinal with  $\lambda \geq \gamma_R + |R| + \aleph_0$  and  $\alpha < \lambda^+$  a limit ordinal. If  $M$  is a  $(\lambda, \alpha)$ -limit model in  $\mathbf{K}^{\text{Th}_{R, \leq}}$  and  $\text{cf}(\alpha) \geq \gamma_R$ , then  $M$  is injective.*

*Proof.* By [Ekl71, Lemma 2] it is enough to show that if  $\mathbb{E} = \{r_\delta x = a_\delta : \delta < \beta\}$  is a system of equations in one free variable  $x$  with  $\beta < \gamma_R$  and  $r_\delta \in R$ ,  $a_\delta \in M$  for every  $\delta < \beta$  and  $\mathbb{E}$  has a solution in an extension of  $M$ , then  $\mathbb{E}$  has a solution in  $M$ .

Let  $\mathbb{E}$  a system of equations as in the previous paragraph,  $M' \in \mathbf{K}_\lambda$  an extension of  $M$  with  $b \in M'$  realizing  $\mathbb{E}$  and  $\{M_i : i < \alpha\}$  a witness to the fact that  $M$  is a  $(\lambda, \alpha)$ -limit model. Since  $\beta < \gamma_R$  and  $\text{cf}(\alpha) \geq \gamma_R$ , there is  $i < \alpha$  such that  $\{a_\delta : \delta < \beta\} \subseteq M_i$ . Since  $M_{i+1}$  is universal over  $M_i$  there is  $f : M' \xrightarrow[M_i]{} M$ . It is clear that  $f(b) \in M$  realizes  $\mathbb{E}$ .  $\square$

Using the above lemma, we can obtain an equivalence in Lemma 3.6

**Corollary 3.8.** *Let  $R$  be a ring and  $\lambda$  an infinite cardinal with  $\lambda \geq |R| + \aleph_0$ .  $\lambda^{<\gamma_R} = \lambda$  if and only if  $\mathbf{K}^{\text{Th}_{R, \leq}}$  is  $\lambda$ -stable.*

*Proof.* The forward direction is Lemma 3.6 and the backward direction follows from the existence of limit models, Lemma 3.7 and Fact 3.5.  $\square$

Doing a similar proof to that of Lemma 3.7 and using the equivalence between saturation and model-homogeneity one can obtain the next result.

**Lemma 3.9.** *Let  $\lambda \geq (|R| + \aleph_0)^+$ . If  $M$  is  $\lambda$ -saturated in  $\mathbf{K}^{\text{Th}_{R, \leq}}$ , then  $M$  is injective.*

Since a ring  $R$  is noetherian if and only  $\gamma_R \leq \aleph_0$  (by Fact 2.19), the next result follows from the results we just obtained in this section.

**Corollary 3.10.** *If  $R$  is a left noetherian ring, then:*

- (1)  $\mathbf{K}^{\text{Th}_{R, \leq}}$  is  $\lambda$ -stable for every  $\lambda \geq |R| + \aleph_0$ .
- (2) There is a  $(\lambda, \alpha)$ -limit model in  $\mathbf{K}^{\text{Th}_{R, \leq}}$  for every  $\alpha < \lambda^+$  and  $\lambda$  with  $\lambda \geq |R| + \aleph_0$ .
- (3) Every limit model in  $\mathbf{K}^{\text{Th}_{R, \leq}}$  is injective.

Moreover, the analogous of [KuMa, 4.8] can also be carried out in this context. Since the proof of the proposition is basically the same as that of [KuMa, 4.8] we omit it.

**Proposition 3.11.** *Assume  $\lambda \geq (|R| + \aleph_0)^+$ . If  $M$  is a  $(\lambda, \omega)$ -limit model in  $\mathbf{K}^{\text{Th}_{R, \leq}}$  and  $N$  is a  $(\lambda, (|R| + \aleph_0)^+)$ -limit model in  $\mathbf{K}^{\text{Th}_{R, \leq}}$ , then  $M \cong N^{(\aleph_0)}$ .*

With this we obtain a new characterization of left noetherian rings via superstability.

**Theorem 3.12.** *For a ring  $R$  the following are equivalent.*

- (1)  $R$  is left noetherian.
- (2) The class of left  $R$ -modules with embeddings is superstable.
- (3) For every  $\lambda \geq |R| + \aleph_0$ , there is a  $\chi \geq \lambda$  such that the class of left  $R$ -modules with embeddings has uniqueness of limit models of cardinality  $\chi$ .
- (4) For every  $\lambda \geq |R| + \aleph_0$ , the class of left  $R$ -modules with embeddings has uniqueness of limit models of cardinality  $\lambda$ .
- (5) For every  $\lambda \geq (|R| + \aleph_0)^+$ , the class of left  $R$ -modules with embeddings has a superlimit of cardinality  $\lambda$ .
- (6) For every  $\lambda \geq |R| + \aleph_0$ , the class of left  $R$ -modules with embeddings is  $\lambda$ -stable.

*Proof.*  $\boxed{(1) \rightarrow (4)}$  Let  $\lambda \geq |R| + \aleph_0$ . By Corollary 3.10.(2) there is  $(\lambda, \alpha)$ -limit models for every  $\alpha < \lambda^+$  limit ordinal. So we only need to show uniqueness of limit models. Let  $M$  and  $N$  two limit models of cardinality  $\lambda$ . By Corollary 3.10.(3)  $M$  and  $N$  are injective and since  $M$  embeds into  $N$  and vice versa by Fact 2.10, it follows from [Bum65] that  $M \cong N$ .

$\boxed{(4) \rightarrow (2) \text{ and } (2) \rightarrow (3)}$  Clear.

$\boxed{(3) \rightarrow (1)}$  We will use the equivalence given in Fact 2.19.(2), so let  $M$  an injective module. Since  $M$  is injective, it is absolutely pure so by Fact 2.21 it is enough to show that  $M$  is  $\Sigma$ -pure-injective. Let  $\chi \geq (||M|| + |R| + \aleph_0)^+$  such that  $\mathbf{K}^{\text{Th}_{R, \leq}}$  has uniqueness of limit models of cardinality  $\chi$  and  $N$  a  $(\chi, (|R| + \aleph_0)^+)$ -limit model such that  $M \leq N$ . By Proposition 3.11  $N^{(\aleph_0)}$  is a  $(\chi, \omega)$ -limit model, then by uniqueness of limit models and Lemma 3.7 (using that  $(|R| + \aleph_0)^+ \geq \gamma_R$ )  $N^{(\aleph_0)}$  is injective. Then by Fact 2.18  $N$  is  $\Sigma$ -injective. Therefore,  $N$  is  $\Sigma$ -pure-injective and since  $M \leq_{pp} N$ , we have by Fact 2.22 that  $M$  is  $\Sigma$ -pure-injective.

$\boxed{(1) \rightarrow (5)}$  Let  $\lambda \geq (|R| + \aleph_0)^+$ . By Corollary 3.10.(1)  $\mathbf{K}^{\text{Th}_{R, \leq}}$  is  $\lambda$ -stable, so let  $M$  a  $(\lambda, (|R| + \aleph_0)^+)$ -limit model. Then by condition (4) together with Fact 2.17.(2)  $M$  is saturated. Then by Fact 2.17.(4) it is enough to show that an increasing chain of saturated models in  $\mathbf{K}_\lambda^{\text{Th}_{R, \leq}}$  is saturated. Let  $\{M_i : i < \delta\}$  an increasing chain of saturated models in  $\mathbf{K}_\lambda^{\text{Th}_{R, \leq}}$  and  $\delta < \lambda^+$ .

Using that  $\gamma_R \leq \aleph_0$  and that  $M_i$  is injective for every  $i < \delta$  (by Lemma 3.9), one can show that  $\bigcup_{i < \delta} M_i$  is an injective module. Moreover,  $\bigcup_{i < \delta} M_i$  embeds into  $M_0$ , because  $M_0$  saturated, and  $M_0$  embeds into  $\bigcup_{i < \delta} M_i$ . Then by [Bum65]  $M_0 \cong \bigcup_{i < \delta} M_i$ . Hence,  $\bigcup_{i < \delta} M_i$  is saturated.

$\boxed{(5) \rightarrow (2)}$  Let  $\lambda \geq |R| + \aleph_0$  and  $\chi = (\lambda^+)^{\gamma_R}$ . By Lemma 3.6  $\mathbf{K}^{\text{Th}_{R, \leq}}$  is  $\chi$ -stable. Therefore there are  $(\chi, \alpha)$ -limit models for every  $\alpha < \chi^+$  limit ordinal. So we only need to show uniqueness of limit models. Let  $M$  and  $N$  two limit models of cardinality  $\chi$ . Let  $L$  a superlimit of size  $\chi$ , then by Fact 2.17.(3)  $M \cong L$  and  $N \cong L$ . Hence  $M \cong N$ .

$\boxed{(1) \rightarrow (6)}$  Corollary 3.10.(1).

$\boxed{(6) \rightarrow (1)}$  Assume for the sake of contradiction that  $R$  is not noetherian, then it follows that  $\gamma_R > \aleph_0$ . Let  $\lambda = \beth_\omega(|R| + \aleph_0)$ . Since  $cf(\lambda) = \omega$ , we have by Königs lemma that  $\lambda^{< \gamma_R} > \lambda$ . Then by Corollary 3.8 it follows that  $\mathbf{K}^{\text{Th}_{R, \leq}}$  is not  $\lambda$ -stable. A contradiction to the hypothesis.  $\square$

This is not the first result where noetherian rings and superstability have been related. As it was mentioned in the introduction, Shelah noticed that superstability of the theory of modules implies that the ring is left noetherian in [Sh75, §8]. The precise equivalence he noticed is similar to that of (6) implies (1) of the above theorem. For a countable  $\omega$ -stable ring a proof is given in [BaMc82, 9.1]. It is clear that Shelah did not know that the two notions are equivalent. Another paper that relates both notions is [GrSh86]. In it, it is shown (for integral domains) that superstability (in the sense of [GrSh86, 1.2]) of the class of torsion divisible modules implies that the ring is noetherian.

**Remark 3.13.** *Compared to [GrVas17, 1.3], the above theorem improves the bounds where the nice property shows up from  $\beth_{(2^{|R| + \aleph_0})^+}$  to  $|R| + \aleph_0$  or  $(|R| + \aleph_0)^+$  in the class of modules with embeddings.*

**Remark 3.14.** *Since  $\mathbf{K}^{\text{Th}_{R, \leq}}$  has amalgamation, joint embedding, no maximal models and is  $(< \aleph_0)$ -tame (by Corollary 3.3). Therefore, we could have simply quoted the main theorem of [GrVas17] and [Vas18] to obtain (5),(6) imply (2) of the above theorem. We decided to provide the proofs for those directions to make the paper more transparent and since the proofs in our case are easier than in the general case.*

## 4. A NEW CHARACTERIZATION OF PURE-SEMISIMPLE RINGS

It is possible to obtain a similar proof for Theorem 4.25 (without condition (7)) as the one presented for Theorem 3.12. The reason we do not do this is because there is a deep theory when one considers theories of modules with pure embeddings. What we will do is to study superstability for any theory of modules with pure embeddings and as a simple corollary we will obtain Theorem 4.25.

In [KuMa, 3.4], it is shown that if  $T$  is a theory of modules then  $\mathbf{K}^T$  is an AEC. Most interesting results do not hold for all AECs, so we will assume, as in [KuMa], the next hypothesis throughout this section.

**Hypothesis 4.1.** *Let  $R$  be a ring and  $T$  a theory of modules with an infinite model such that:*

- (1)  $\mathbf{K}^T$  has joint embedding.
- (2)  $\mathbf{K}^T$  has amalgamation.

These may seem like *ad hoc* hypothesis, but there are many natural theories satisfying them. This is the case if  $T$  is a complete theory, but many other examples are given in [KuMa, 3.10]. For the proof of the main theorem, we will only use that  $\mathbf{K}^{\text{Th}_R}$  satisfies the above hypothesis, which clearly does.

Since the theory of modules has *pp*-quantifier elimination (see for example [Zie84, 1.1]), one can show the following.

**Fact 4.2** ([KuMa, 3.14]). *Let  $M, N_1, N_2 \in \mathbf{K}^T$ ,  $M \leq_{pp} N_1, N_2$ ,  $\bar{b}_1 \in N_1^{<\omega}$  and  $\bar{b}_2 \in N_2^{<\omega}$ . Then:*

$$\mathbf{gtp}(\bar{b}_1/M; N_1) = \mathbf{gtp}(\bar{b}_2/M; N_2) \text{ if and only if } pp(\bar{b}_1/M, N_1) = pp(\bar{b}_2/M, N_2).$$

Moreover, if  $N_1 \equiv N_2$  one can substitute the *pp*-types by the first-order types ([KuMa, 3.13]).

**4.1. The theory  $\tilde{T}$ .** In [KuMa, 3.16] it is shown that  $\mathbf{K}^T$  is  $\lambda$ -stable if  $\lambda^{|T|} = \lambda$ . Then it follows from Fact 2.11 that there exist limit models of cardinality  $\lambda$  in  $\mathbf{K}^T$  for  $\lambda$ 's such that  $\lambda^{|T|} = \lambda$ . More importantly and key to the naturality of the theory we will introduce in this section is the following result.

**Fact 4.3** ([KuMa, 4.3]). *If  $M$  and  $N$  are limit models in  $\mathbf{K}^T$ , then  $M$  and  $N$  are elementary equivalent.*

Let us introduce the main notion of this subsection.

**Notation 4.4.** *For  $T$  a theory of modules, let  $\tilde{M}_T$  be the  $(2^{|T|}, \omega)$ -limit model of  $\mathbf{K}^T$  and  $\tilde{T} = \text{Th}(\tilde{M}_T)$ .*

It is natural to ask which structures of  $\mathbf{K}^T$  satisfy the complete first-order theory  $\tilde{T}$ . It follows from Fact 4.3 that limit models do, we record this for future reference.

**Corollary 4.5.** *If  $M$  is a limit model in  $\mathbf{K}^T$ , then  $M$  is a model of  $\tilde{T}$ .*

The next lemma gives another class of structures satisfying  $\tilde{T}$ .

**Lemma 4.6.** *Let  $\lambda \geq |T|^+$ . If  $M$  is a  $\lambda$ -saturated model, then  $M$  is a model of  $\tilde{T}$ .*

*Proof sketch.* The proof is similar to that of [KuMa, 4.3], by using the equivalence between  $\lambda$ -saturation and  $\lambda$ -model-homogeneity.  $\square$

Moreover,  $\tilde{T}$  is closed upward under pure extensions.

**Lemma 4.7.** *If  $M$  is a model of  $\tilde{T}$  and  $M \leq_{pp} N$ , then  $M \preceq N$ . In particular,  $N$  is a model of  $\tilde{T}$ .*

*Proof.* Let  $\lambda = \|N\|^{|T|}$ , then by [KuMa, 3.16]  $\mathbf{K}^T$  is  $\lambda$ -stable. So let  $N^* \in \mathbf{K}_\lambda^T$  such that  $N \leq_{pp} N^*$  and  $N^*$  is a  $(\lambda, \omega)$ -limit model. Since  $\tilde{T}$  is a complete theory and by Corollary 4.5,  $N^* \equiv M$ . Then  $M \preceq N \preceq N^*$  by [Pre88, 2.25]. Whence  $M \preceq N$  and  $N$  is a model of  $\tilde{T}$ .  $\square$

The next result shows the naturality of  $\tilde{T}$ , the proof is similar to that of the above lemma so we omit it.

**Corollary 4.8.** *If  $M \in \mathbf{K}^T$ , then there is  $N$  a model of  $\tilde{T}$  such that  $M \leq_{pp} N$ . Moreover, if  $T'$  is a complete first-order theory with this property, then  $\tilde{T} = T'$ .*

The following lemmas show that there is a close relationship between the class  $\mathbf{K}^T$  and the first-order theory  $\tilde{T}$ . This is useful since complete first-order theories of modules are very well understood (see for instance [Pre88]).

**Lemma 4.9.** *For  $T$  a theory of modules and  $\lambda \geq |T|$ , the following are equivalent.*

- (1)  $\tilde{T}$  is  $\lambda$ -stable.
- (2)  $\mathbf{K}^T$  is  $\lambda$ -stable.

*Proof.*  $\boxed{\Rightarrow}$  Let  $M \in \mathbf{K}_\lambda^T$  and  $\{p_i : i < \alpha\}$  an enumeration without repetitions of  $\mathbf{gS}(M)$ . Fix  $N \in \mathbf{K}_\lambda^T$  a  $(\lambda, \omega)$ -limit model over  $M$ . Then there is  $\{a_i : i < \alpha\} \subseteq N$  such that  $p_i = \mathbf{gtp}(a_i/M; N)$  for every  $i < \alpha$ .

Let  $\Phi : \mathbf{gS}(M) \rightarrow S^{Th(N)}(M)$  be defined by  $\phi(\mathbf{gtp}(a_i/M; N)) = tp(a_i/M, N)$ . By Fact 4.2 it follows that  $\Phi$  is a well-defined injective function, so  $|\mathbf{gS}(M)| \leq |S^{Th(N)}(M)|$ . Finally, since  $N$  is a model of  $\tilde{T}$  by Corollary 4.5 and by hypothesis  $\tilde{T}$  is  $\lambda$ -stable, it follows that  $|S^{Th(N)}(M)| \leq \|M\|$ . Hence  $|\mathbf{gS}(M)| \leq \|M\|$ .

$\boxed{\Leftarrow}$  Let  $A \subseteq N$  of size  $\lambda$  with  $N \models \tilde{T}$  and  $N$   $\lambda^+$ -saturated in  $\tilde{T}$ . Let  $\{p_i : i < \alpha\}$  an enumeration without repetitions of  $S^{\tilde{T}}(A)$ . Let  $\{a_i : i < \alpha\} \subseteq N$  such that  $p_i = tp(a_i/A, N)$  for every  $i < \alpha$ .

Let  $M$  be the structure obtained by applying downward Löwenheim-Skolem-Tarski to  $A$  in  $N$ , observe that  $\|M\| = \lambda$  because  $\lambda \geq |T|$ . Let  $\Psi : S^{\tilde{T}}(A) \rightarrow \mathbf{gS}(M)$  be defined by  $\phi(tp(a_i/A, N)) = \mathbf{gtp}(a_i/M; N)$ . By Fact 4.2 it follows that  $\Psi$  is a well-defined injective function and doing a similar argument to the one above it follows that  $|S^{\tilde{T}}(A)| \leq \|M\| = |A|$ .  $\square$

**Lemma 4.10.** *For  $T$  a theory of modules and  $\lambda \geq |T|^+$ , the following are equivalent.*

- (1)  $M$  is a model of  $\tilde{T}$  and  $M$  is  $\lambda$ -saturated in  $\tilde{T}$ .
- (2)  $M$  is  $\lambda$ -saturated in  $\mathbf{K}^T$ .

*Proof.*  $\boxed{\Rightarrow}$  Let  $L \in \mathbf{K}^T$ ,  $L \leq_{pp} M$ ,  $\|L\| < \lambda$  and  $p \in \mathbf{gS}(L)$ . Let  $L^*$  a  $(\|L\|^{|T|}, \omega)$ -limit model such that  $L \leq_{pp} L^*$  and  $a \in L^*$  with  $p = \mathbf{gtp}(a/L; L^*)$ .

Realize that  $L^*$  is a model of  $\tilde{T}$  by Fact 4.3, so  $tp(a/L, L^*) \in S^{\tilde{T}}(L)$ . Then since  $M$  is  $\lambda$ -saturated in  $\tilde{T}$  there is  $b \in M$  such that  $tp(a/L, L^*) = tp(b/L, M)$ . Therefore, by  $L^* \equiv M$  and Fact 4.2,  $\mathbf{gtp}(a/L; L^*) = \mathbf{gtp}(b/L; M)$ .

$\boxed{\Leftarrow}$  By Lemma 4.6  $M$  is a model of  $\tilde{T}$ . Let  $A \subseteq M$  and  $p \in S^{\tilde{T}}(A)$ . Let  $N$  elementary extension of  $M$  and  $a \in N$  such that  $p = tp(a/A, N)$ . Let  $M^*$  be the structure obtained by applying downward Löwenheim-Skolem-Tarski to  $A$  in  $M$ , observe that  $\|M^*\| < \lambda$  because  $\lambda \geq |T|^+$ .

Realize that  $M^* \leq_{pp} M \leq_{pp} N$ , so  $\mathbf{gtp}(a/M^*; N) \in \mathbf{gS}(M^*)$ . Then since  $M$  is  $\lambda$ -saturated in  $\mathbf{K}^T$ , there is  $b \in M$  such that  $\mathbf{gtp}(a/M^*; N) = \mathbf{gtp}(b/M^*; M)$ . Therefore, by  $M \equiv N$  and Fact 4.2,  $tp(a/M^*, N) = tp(b/M^*, M)$ . Hence  $b \in M$  realizes  $p$ .  $\square$

The following result was pointed out to us by an anonymous referee.

**Lemma 4.11.** *For  $T$  a theory of modules,  $\lambda \geq |T|$  and  $\alpha < \lambda^+$  a limit ordinal, the following are equivalent.*

- (1)  $M$  is a  $(\lambda, \alpha)$ -limit model in  $(\text{Mod}(\tilde{T}), \preceq)$ .
- (2)  $M$  is a  $(\lambda, \alpha)$ -limit model in  $\mathbf{K}^T$ .

*Proof.*  $\boxed{\rightarrow}$  Let  $\{M_i : i < \alpha\}$  a witness to the fact that  $M$  is a  $(\lambda, \alpha)$ -limit model in  $(\text{Mod}(\tilde{T}), \preceq)$ . Observe that  $\{M_i : i < \alpha\}$  is chain of models in  $\mathbf{K}^T$ , so it is enough to show that  $M_{i+1}$  is universal over  $M_i$  for every  $i < \alpha$  in  $\mathbf{K}^T$ . This follows easily from Lemma 4.7.

$\boxed{\leftarrow}$  Let  $\{M_i : i < \alpha\}$  a witness to the fact that  $M$  is a  $(\lambda, \alpha)$ -limit model in  $\mathbf{K}^T$ .  $\mathbf{K}^T$  is  $\lambda$ -stable, by Fact 2.11, so let  $N$  a  $(\lambda, \omega)$ -limit model over  $M_0$ . Since  $M_1$  is universal over  $M_0$ , there is  $f : N \xrightarrow{M_0} M_1$ . Then by Lemma 4.7 it follows that  $\{M_i : 0 < i < \alpha\}$  is an elementary chain of models of  $\tilde{T}$ . Using Lemma 4.7 once again, one can show that  $M_{i+1}$  is universal over  $M_i$  for every  $i < \alpha$  in  $(\text{Mod}(\tilde{T}), \preceq)$ . Hence  $\{M_i : 0 < i < \alpha\}$  is a witness to the fact that  $M$  is a  $(\lambda, \alpha)$ -limit model in  $(\text{Mod}(\tilde{T}), \preceq)$ .  $\square$

Given  $\phi, \psi$   $pp$ -formulas in one free variable such that  $\mathbf{Th}_R \vdash \psi \rightarrow \phi$  and  $M$  a module,  $\text{Inv}(M, \phi, \psi) = [\phi(M) : \psi(M)]$ .

**Lemma 4.12.** *Let  $T$  a theory of modules. If  $\mathbf{K}^T$  is closed under direct sums, then  $\tilde{T}$  is closed under direct sums.*

*Proof.* Recall that  $\tilde{M}_T$  is the  $(2^{|T|}, \omega)$ -limit model of  $\mathbf{K}^T$ .

$\boxed{\text{Claim}}$   $\text{Inv}(\tilde{M}_T, \phi, \psi) = 1$  or  $\infty$  for every  $\phi, \psi$   $pp$ -formulas in one free variable such that  $\mathbf{Th}_R \vdash \psi \rightarrow \phi$ .

Proof of Claim: Let  $\phi, \psi$   $pp$ -formulas such that  $\mathbf{Th}_R \vdash \psi \rightarrow \phi$  and assume for the sake of contradiction that  $\text{Inv}(\tilde{M}_T, \phi, \psi) = k > 1$  for  $k \in \mathbb{N}$ . Since  $\mathbf{K}^T$  is closed under direct sums,  $\tilde{M}_T \oplus \tilde{M}_T \in \mathbf{K}^T$  and by Fact 2.10 there is  $f : \tilde{M}_T \oplus \tilde{M}_T \rightarrow \tilde{M}_T$  pure embedding. Then:

$$k^2 = \text{Inv}(\tilde{M}_T \oplus \tilde{M}_T, \phi, \psi) = \text{Inv}(f[\tilde{M}_T \oplus \tilde{M}_T], \phi, \psi) \leq \text{Inv}(\tilde{M}_T, \phi, \psi) = k$$

The first equality and last inequality follow from [Pre88, 2.23]. Clearly the above inequality gives us a contradiction.  $\uparrow_{\text{Claim}}$

Let  $N_1, N_2$  models of  $\tilde{T}$ . To show that  $N_1 \oplus N_2$  is a model of  $\tilde{T}$ , by [Pre88, 2.18] it is enough to show that  $\text{Inv}(N_1 \oplus N_2, \phi, \psi) = \text{Inv}(\tilde{M}_T, \phi, \psi)$  for every  $\phi, \psi$   $pp$ -formulas in one free variable such that  $\mathbf{Th}_R \vdash \psi \rightarrow \phi$ . Since  $\text{Inv}(N_1 \oplus N_2, \phi, \psi) = \text{Inv}(N_1, \phi, \psi)\text{Inv}(N_2, \phi, \psi)$  (by [Pre88, 2.23]), the result follows from the above claim.  $\square$

**4.2. Superstability in classes closed under direct sums.** In this section we will characterize superstability in classes of modules with pure embeddings closed under direct sums. As a direct consequence of this we will obtain several new characterizations of pure-semisimple rings.

In [KuMa, §4] limit models on classes of the form  $\mathbf{K}^T$  were studied. Below we record the two assertions we will use in this paper.

**Fact 4.13** ([KuMa, 4.4]). *Assume  $\lambda \geq |T|^+$ . If  $M$  is a  $(\lambda, \alpha)$ -limit model in  $\mathbf{K}^T$  and  $\text{cf}(\alpha) \geq |T|^+$ , then  $M$  is pure-injective.*

**Fact 4.14** ([KuMa, 4.8]). *Assume  $\lambda \geq |T|^+$  and  $\mathbf{K}^T$  is closed under direct sums. If  $M$  is a  $(\lambda, \omega)$ -limit model and  $N$  is a  $(\lambda, |T|^+)$ -limit model, then  $M \cong N^{(\aleph_0)}$ .*

With this we are ready to obtain the next result.

**Lemma 4.15.** *Assume  $\mathbf{K}^T$  is closed under direct sums. If there exists  $\mu \geq |T|^+$  such that  $\mathbf{K}^T$  has uniqueness of limit models of cardinality  $\mu$ , then every  $(\lambda, \alpha)$ -limit model is  $\Sigma$ -pure-injective for every  $\lambda \geq |T|$  and  $\alpha < \lambda^+$  limit ordinal.*

*Proof.* Let  $M \in \mathbf{K}_\lambda^T$  a  $(\lambda, \alpha)$ -limit model. Fix  $N \in \mathbf{K}_\mu^T$  a  $(\mu, \omega)$ -limit model and  $N^* \in \mathbf{K}_\mu^T$  a  $(\mu, |T|^+)$ -limit model. By Fact 4.14 we have that  $N \cong (N^*)^{(\omega)}$ .

Then by uniqueness of limit models of size  $\mu$  we have that  $N \cong N^*$ . Hence  $(N^*)^{(\omega)} \cong N^*$ . Moreover,  $N^*$  is pure-injective by Fact 4.13. Therefore,  $N^*$  is  $\Sigma$ -pure-injective.

Finally, observe that by Fact 4.3  $N^* \equiv M$ , hence  $M$  is  $\Sigma$ -pure-injective.  $\square$

Observe that in the above proof we did not use the full-strength of uniqueness of limit models of size  $\mu$ , but the weaker statement that the  $(\mu, \omega)$ -limit model is isomorphic to the  $(\mu, |T|^+)$ -limit model. We record it as a corollary for future reference.

**Corollary 4.16.** *Assume  $\mathbf{K}^T$  is closed under direct sums. If there exists  $\mu \geq |T|^+$  such that the  $(\mu, \omega)$ -limit model is isomorphic to the  $(\mu, |T|^+)$ -limit model, then every  $(\lambda, \alpha)$ -limit model is  $\Sigma$ -pure-injective for every  $\lambda \geq |T|$  and  $\alpha < \lambda^+$  limit ordinal.*

**Lemma 4.17.** *Assume  $\mathbf{K}^T$  is closed under direct sums. If there exists  $\mu \geq |T|^+$  such that  $\mathbf{K}^T$  has uniqueness of limit models of cardinality  $\mu$ , then  $\mathbf{K}^T$  is  $\lambda$ -stable for every  $\lambda \geq |T|$ .*

*Proof.* Since  $\mathbf{K}^T$  has uniqueness of limit models of size  $\mu$ , by Lemma 4.15  $\tilde{M}_T$  is  $\Sigma$ -pure-injective. Then by Fact 2.23  $Th(\tilde{M}_T) = \tilde{T}$  is  $\lambda$ -stable for every  $\lambda \geq |T|$ . Therefore, it follows from Lemma 4.9 that  $\mathbf{K}^T$  is  $\lambda$ -stable for every  $\lambda \geq |T|$ .  $\square$

The next result is easy to prove, but due to its importance we record it.

**Corollary 4.18.** *If  $M, N$  are limit models and  $M, N$  are pure-injective, then  $M \cong N$ .*

*Proof.* Since  $M, N$  are limit models, it follows from Fact 2.10 that there are  $f : M \rightarrow N$  and  $g : N \rightarrow M$  pure embeddings. Then by Fact 2.26 and the hypothesis that  $M, N$  are pure-injective, we conclude that  $M \cong N$ .  $\square$

The next lemma is one of the key assertions of the section. In it we show that if the class is closed under direct sums, uniqueness of limit models in *one* cardinal implies uniqueness of limit models in *all* cardinals.

**Lemma 4.19.** *Assume  $\mathbf{K}^T$  is closed under direct sums. The following are equivalent.*

- (1) *For every  $\lambda \geq |T|$ ,  $\mathbf{K}^T$  has uniqueness of limit models of cardinality  $\lambda$ .*
- (2)  *$\mathbf{K}^T$  is superstable.*
- (3) *There exists  $\lambda \geq |T|^+$  such that  $\mathbf{K}^T$  has uniqueness of limit models of cardinality  $\lambda$ .*

*Proof.* (1) implies (2) and (2) implies (3) are clear, so we show (3) implies (1).

Let  $\lambda \geq |T|$ . By Lemma 4.17 it follows that  $\mathbf{K}^T$  is  $\lambda$ -stable. Hence for every  $\alpha < \lambda^+$  there is a  $(\lambda, \alpha)$ -limit model by Fact 2.11. So we only need to show uniqueness of limit models. Let  $M$  and  $N$  two limit models of cardinality  $\lambda$ . By Lemma 4.15 we have that  $M$  and  $N$  are both pure-injective modules. Therefore, it follows from the above corollary that  $M \cong N$ .  $\square$

We will give several additional equivalent conditions to the ones of Lemma 4.19, but before we do that let us characterize superlimits in classes of modules.

**Lemma 4.20.** *Assume  $\mathbf{K}^T$  is  $\lambda$ -stable. If  $M$  is a superlimit of size  $\lambda$ , then  $M$  is pure-injective. Moreover, if  $\mathbf{K}^T$  is closed under direct sums, then  $M$  is  $\Sigma$ -pure-injective.*

*Proof sketch.* By Fact 2.17.(3)  $M$  is isomorphic to every  $(\lambda, \alpha)$ -limit model for  $\alpha < \lambda^+$  limit ordinal. Then by Fact 4.13  $M$  is pure-injective. For the moreover part, observe that the existence of a superlimit and the stability assumption imply uniqueness of limit models. Therefore, by Lemma 4.15,  $M$  is  $\Sigma$ -pure-injective.  $\square$

The following lemma can be proven using a similar technique to Fact 2.17.(2) and using [KuMa, 4.6].

**Lemma 4.21.** *If  $M$  is a  $(\lambda, |T|^+)$ -limit model, then  $M$  is saturated in  $\mathbf{K}^T$ .*

The following theorem characterizes superstability in classes of modules closed under direct sums.

**Theorem 4.22.** *Assume  $\mathbf{K}^T$  satisfies Hypothesis 4.1 and is closed under direct sums. The following are equivalent.*

- (1)  $\mathbf{K}^T$  is superstable.
- (2) There exists  $\lambda \geq |T|^+$  such that  $\mathbf{K}^T$  has uniqueness of limit models of cardinality  $\lambda$ .
- (3) For every  $\lambda \geq |T|$ ,  $\mathbf{K}^T$  has uniqueness of limit models of cardinality  $\lambda$ .
- (4) For every  $\lambda \geq |T|^+$ ,  $\mathbf{K}^T$  has a superlimit of cardinality  $\lambda$ .
- (5) For every  $\lambda \geq |T|$ ,  $\mathbf{K}^T$  is  $\lambda$ -stable.
- (6) For every  $\lambda \geq |T|^+$ , an increasing chain of  $\lambda$ -saturated models in  $\mathbf{K}^T$  is  $\lambda$ -saturated in  $\mathbf{K}^T$ .

*Proof.*  $(1) \leftrightarrow (2) \leftrightarrow (3)$  By Lemma 4.19.

$(2) \rightarrow (5)$  By Lemma 4.17.

$(5) \rightarrow (2)$  By Lemma 4.9  $\tilde{T}$  is  $\lambda$ -stable for every  $\lambda \geq |T|$ . Then by [Pre88, 3.1] every model of  $\tilde{T}$  is  $\Sigma$ -pure-injective. Let  $\lambda = |T|^+$ . By  $\lambda$ -stability and Fact 2.11 there are  $(\lambda, \alpha)$ -limit models for every  $\alpha < \lambda^+$  limit ordinal. The uniqueness of limit models of cardinality  $\lambda$  follows from the fact that limit models are pure-injective, since they are models of  $\tilde{T}$  by Corollary 4.5, and by Corollary 4.18.

$(5) \rightarrow (6)$  Let  $\{M_i : i < \delta\} \subseteq \mathbf{K}^T$  an increasing chain of  $\lambda$ -saturated models. By Lemma 4.10 every  $M_i$  is a model of  $\tilde{T}$  and  $\lambda$ -saturated in  $\tilde{T}$ . Moreover, by Lemma 4.7, for every  $i < j$  we have that  $M_i \preceq M_j$ . Therefore,  $\{M_i : i < \delta\}$  is an increasing chain of  $\lambda$ -saturated models in  $\tilde{T}$ .

Then by hypothesis and Lemma 4.9  $\tilde{T}$  is superstable as a first-order theory. Hence, by [Har75],  $\bigcup_{i < \delta} M_i$  is a  $\lambda$ -saturated model of  $\tilde{T}$ . Therefore, by Lemma 4.10,  $\bigcup_{i < \delta} M_i$  is  $\lambda$ -saturated.

$(6) \rightarrow (2)$  Let  $\lambda = 2^{|T|}$  and  $M$  a  $(2^{|T|}, |T|^+)$ -limit model. By Fact 4.13  $M$  is pure-injective and by Lemma 4.21  $M$  is saturated. Consider  $\{M^n : n \in \omega \setminus \{0\}\}$  an increasing chain in  $\mathbf{K}_{2^{|T|}}$  where  $M^n$  denotes  $n$ -many direct copies of  $M$ . Observe that each  $M^n$  is pure-injective (because pure-injectivity is closed under finite direct sums) and that there are pure embeddings between  $M^n$  and  $M$  and vice versa by Fact 2.10. Therefore, by Fact 2.26,  $M \cong M^n$  for every  $n \in \omega \setminus \{0\}$ . So in particular,  $M^n$  is  $\|M\|$ -saturated for every  $n \in \omega \setminus \{0\}$ .

Then by hypothesis  $M^{(\aleph_0)}$  is  $\|M\|$ -saturated, so  $M^{(\aleph_0)}$  is saturated. Since saturated models of the same cardinality are isomorphic,  $M^{(\aleph_0)} \cong M$ . On the other hand, by Fact 4.14,  $M^{(\aleph_0)}$  is the  $(2^{|T|}, \omega)$ -limit model. Then by Corollary 4.16 every limit model is  $\Sigma$ -pure-injective. Therefore, by Corollary 4.18,  $\mathbf{K}^T$  has uniqueness of limit models of size  $2^{|T|}$ .

$(4) \rightarrow (2)$  Similar to  $(5) \rightarrow (2)$  of Theorem 3.12.

$\boxed{(3) \rightarrow (4)}$  Let  $\lambda \geq |T|^+$ . By condition (5)  $\mathbf{K}^T$  is  $\lambda$ -stable, so let  $M$  a  $(\lambda, |T|^+)$ -limit model. By Lemma 4.21  $M$  is  $\|M\|$ -saturated. Moreover, by condition (6) any increasing chain of  $\|M\|$ -saturated models is  $\|M\|$ -saturated. Therefore it follows from Fact 2.17.(3) that there is a superlimit of cardinality  $\lambda$ .  $\square$

**Remark 4.23.** In [GrVas17, 1.3] cardinals  $\mu_\ell$  for  $\ell \in \{1, \dots, 7\}$  were introduced. In the introduction of [GrVas17] it is asked if it is possible to calculate the values of the  $\mu_\ell$ 's for certain AECs. The above lemma shows that in classes of modules satisfying Hypothesis 4.1 and closed under direct sums  $\mu_3 = \mu_7 = |T|$  and  $\mu_4, \mu_5 \leq |T|^+$ . We did not calculate the values of  $\mu_1, \mu_2$  and  $\mu_6$  since they measure properties that are more technical than the ones presented above and which we have not introduced. We hope to study those properties in future work.

**Remark 4.24.** Let  $T$  a theory of modules such that Hypothesis 4.1 holds. Then by Fact 4.2  $\mathbf{K}^T$  is  $(\aleph_0)$ -tame and by hypothesis  $\mathbf{K}^T$  has amalgamation, joint embedding and no maximal models. Therefore, we could have simply quoted [GrVas17, 1.3] to obtain (4), (5), (6) imply (2) of the above theorem. We decided to provide the proofs for those directions to make the paper more transparent and since the proofs in our case are easier than in the general case. An important difference between our methods and those of [GrVas17] and [Vas18] is that the results of Grossberg and Vasey do not use the hypothesis that the class is closed under direct sums, but only Hypothesis 4.1. We will come back to this in Subsection 4.4.

**4.3. Pure-semisimple rings.** The next theorem gives many equivalent conditions to the notion of pure-semisimple ring and connects it to superstability.

**Theorem 4.25.** For a ring  $R$  the following are equivalent.

- (1)  $R$  is left pure-semisimple.
- (2)  $\mathbf{K}^{\text{Th}_R}$  is superstable.
- (3) There exists  $\lambda \geq (|R| + \aleph_0)^+$  such that  $\mathbf{K}^{\text{Th}_R}$  has uniqueness of limit models of cardinality  $\lambda$ .
- (4) For every  $\lambda \geq |R| + \aleph_0$ ,  $\mathbf{K}^{\text{Th}_R}$  has uniqueness of limit models of cardinality  $\lambda$ .
- (5) For every  $\lambda \geq (|R| + \aleph_0)^+$ ,  $\mathbf{K}^{\text{Th}_R}$  has a superlimit of cardinality  $\lambda$ .
- (6) For every  $\lambda \geq |R| + \aleph_0$ ,  $\mathbf{K}^{\text{Th}_R}$  is  $\lambda$ -stable.
- (7) For every  $\lambda \geq (|R| + \aleph_0)^+$ , an increasing chain of  $\lambda$ -saturated models in  $\mathbf{K}^{\text{Th}_R}$  is  $\lambda$ -saturated in  $\mathbf{K}^{\text{Th}_R}$ .

*Proof.* The equivalences between (2), (3), (4), (5), (6) and (7) follow from Theorem 4.22 since  $\mathbf{K}^{\text{Th}_R}$  satisfies Hypothesis 4.1 and is closed under direct sums. So we only need to show that (1) is equivalent to the other conditions.

$\boxed{(1) \rightarrow (3)}$  Let  $\lambda = 2^{|R| + \aleph_0}$ . Observe that  $\mathbf{K}^{\text{Th}_R}$  is  $\lambda$ -stable by [KuMa, 3.16]. Therefore there are  $(\lambda, \alpha)$ -limit models for every  $\alpha < \lambda^+$  limit ordinal. So we only need to show uniqueness of limit models. Let  $M$  and  $N$  two limit models of cardinality  $\lambda$ . Since  $M$  and  $N$  are pure-injective by the assumption on the ring, it follows from Corollary 4.18 that  $M \cong N$ . Therefore,  $\mathbf{K}^{\text{Th}_R}$  has uniqueness of limit models of cardinality  $\lambda$ .

$\boxed{(4) \rightarrow (1)}$  We will use the equivalence given in Fact 2.25.(2). Let  $M$  an  $R$ -module, we show that  $M$  is  $\Sigma$ -pure-injective. Let  $\lambda = \|M\|^{>|R| + \aleph_0}$ .

By [KuMa, 3.16]  $\mathbf{K}^{\text{Th}_R}$  is  $\lambda$ -stable, so let  $N$  a  $(\lambda, \omega)$ -limit model such that  $M \leq_{pp} N$ . Then since  $\mathbf{K}^{\text{Th}_R}$  has uniqueness of limit models of size  $\lambda$ , by Lemma 4.15  $N$  is  $\Sigma$ -pure-injective. Therefore, by Fact 2.22 it follows that  $M$  is  $\Sigma$ -pure-injective. Therefore,  $R$  is a left pure-semisimple ring.  $\square$

One can add one more equivalent condition to the above theorem.

**Lemma 4.26.** *For a ring  $R$  the following are equivalent.*

- (1)  $R$  is left pure-semisimple.
- (3) There exists  $\lambda \geq (|R| + \aleph_0)^+$  such that  $\mathbf{K}^{Th_R}$  has uniqueness of limit models of cardinality  $\lambda$ .
- (8) For all theory of modules  $T$  and for all  $\lambda \geq |R| + \aleph_0$ , if  $\mathbf{K}^T$  satisfies Hypothesis 4.1 and it is closed under direct sums, then  $\mathbf{K}^T$  has uniqueness of limit models of cardinality  $\lambda$ .

*Proof.* The proof of (1) implies (8) is similar to the proof of (1) implies (3) of the previous theorem. Moreover, it is clear that (8) implies (3).  $\square$

We think that Theorem 3.12 and Theorem 4.25 hint to the the fact that limit models and superstability could shed light in the understanding of algebraic concepts. It also provides further evidence of the naturality of the notion of superstability.

**4.4. Superstable classes.** In this section we will characterize superstability in classes of modules without assuming that the class is closed under direct sums. As in previous subsections we assume Hypothesis 4.1. In this section we assume the reader has some familiarity with first-order model theory.

Recall that given  $T'$  a complete first-order theory,  $\kappa(T')$  is the least cardinal such that every type of  $T'$  does not fork over a set of size less than  $\kappa(T')$ . In this case, nonforking refers to first-order nonforking. Since it is well-known that  $\kappa(T') \leq |T'|^+$  for stable theories, the following improves [KuMa, 4.6].

**Lemma 4.27.** *Let  $\lambda \geq |T|$  and  $\alpha, \beta < \lambda^+$ . If  $M$  is a  $(\lambda, \alpha)$ -limit model in  $\mathbf{K}^T$ ,  $N$  is a  $(\lambda, \beta)$ -limit model in  $\mathbf{K}^T$  and  $cf(\alpha), cf(\beta) \geq \kappa(\tilde{T})$ , then  $M \cong N$ .*

*Proof.* By Lemma 4.11  $M$  is a  $(\lambda, \alpha)$ -limit model in  $(Mod(\tilde{T}), \preceq)$  and  $N$  is a  $(\lambda, \beta)$ -limit model in  $(Mod(\tilde{T}), \preceq)$ . Then it follows from [GVV16, 1.6] that  $M$  and  $N$  are both saturated models of cardinality  $\lambda$  in  $\tilde{T}$ . Therefore, we conclude that  $M \cong N$ .  $\square$

Recall the following notions introduced in [Vas18, §2]. Given an AEC  $\mathbf{K}$  and  $\mu$  a cardinal,  $\text{Stab}(\mathbf{K}) = \{\mu : \mathbf{K} \text{ is } \mu\text{-stable}\}$  and  $\underline{\kappa}(\mathbf{K}_\mu, \leq_{\mathbf{K}}^{\text{univ}})$  is the set of regular cardinals  $\chi$  such that whenever  $\{M_i : i < \chi\}$  is an increasing chain in  $\mathbf{K}_\mu$  with  $M_{i+1}$  is universal over  $M_i$  for every  $i < \chi$  and  $p \in \mathbf{gS}(\bigcup_{i < \chi} M_i)$ , then there is  $i < \chi$  such that  $p$  does not split over  $M_i$ . Since we will not use the notion of splitting we will not introduce it, it is somehow similar to first-order splitting and the interested reader can consult the definition in [Vas18, 2.3].

In [KuMa, 4.11] it is asked to describe the spectrum of limit models for classes of the form  $\mathbf{K}^T$  where  $T$  is a theory of modules. The case when the ring is countable is studied in [KuMa, 4.12]. The next result provides a partial solution to the question.

**Theorem 4.28.** *Let  $\lambda \geq |T|^+$  a regular cardinal. Let  $M$  a  $(\lambda, \alpha)$ -limit model in  $\mathbf{K}^T$ , then:*

- (1) *If  $cf(\alpha) \geq \kappa(\tilde{T})$ , then  $M$  is isomorphic to the  $(\lambda, \lambda)$ -limit model.*
- (2) *If  $cf(\alpha) < \kappa(\tilde{T})$ , then  $M$  is not isomorphic to the  $(\lambda, \lambda)$ -limit model.*

*Proof.* (1) follows from Lemma 4.27 and the fact that  $\kappa(\tilde{T}) \leq |T|^+$ . So we prove (2). Assume for the sake of contradiction that  $M$  is isomorphic to the  $(\lambda, \lambda)$ -limit model in  $\mathbf{K}^T$  and let  $\tilde{\mathbf{K}} := (Mod(\tilde{T}), \preceq)$ .

Since  $cf(\alpha) < \kappa(\tilde{T})$ , by [Vas18, 4.8]  $cf(\alpha) \notin \underline{\chi}(\tilde{\mathbf{K}}) = \bigcup_{\mu \in \text{Stab}(\tilde{\mathbf{K}})} \underline{\kappa}(\tilde{\mathbf{K}}_\mu, \leq_{\tilde{\mathbf{K}}}^{\text{univ}})$ . Since  $M$  is a limit model in  $\mathbf{K}^T$ , by Fact 2.11,  $\mathbf{K}^T$  is  $\lambda$ -stable so by Lemma 4.9  $\tilde{\mathbf{K}}$  is  $\lambda$ -stable. Hence  $cf(\alpha) \notin \underline{\kappa}(\tilde{\mathbf{K}}_\lambda, \leq_{\tilde{\mathbf{K}}}^{\text{univ}})$ . Then by definition of  $\underline{\kappa}$  there is  $\{L_i : i < cf(\alpha)\}$  an increasing chain in

$\tilde{\mathbf{K}}_\lambda$  with  $L_{i+1}$  universal over  $L_i$  and  $p \in \mathbf{gS}(\bigcup_{i < cf(\alpha)} L_i)$  such that  $p$  splits over  $L_i$  in  $\tilde{\mathbf{K}}$  for every  $i < cf(\alpha)$ .

Observe that  $L = \bigcup_{i < cf(\alpha)} L_i$  is a  $(\lambda, cf(\alpha))$ -limit model in  $\mathbf{K}^T$  by Lemma 4.11. Then doing a back-and-forth argument  $L \cong M$ . And since by hypothesis  $M$  is isomorphic to the  $(\lambda, \lambda)$ -limit model,  $L$  is isomorphic to it. By Fact 2.17.(1) it follows that  $L$  is a  $\lambda$ -saturated model in  $\mathbf{K}^T$ . So by Lemma 4.10  $L$  is  $\lambda$ -saturated in  $\tilde{\mathbf{K}}$ . Then by [Vas18, 4.12] there is an  $i < cf(\alpha)$  such that  $p$  does not split over  $L_i$  in  $\tilde{\mathbf{K}}$ . This contradicts the choice of the  $L'_i$ s.  $\square$

Moreover, the result of the above theorem gives a positive solution above  $|T|^+$  to Conjecture 2 of [BoVan] in the case when  $T$  is a theory of modules satisfying Hypothesis 4.1. This provides further evidence for Conjecture 2 of [BoVan].

**Corollary 4.29.** *Let  $T$  a theory of modules such that  $\mathbf{K}^T$  satisfies Hypothesis 4.1. Assume that  $\lambda \geq |T|^+$  is a regular cardinal such that  $\mathbf{K}^T$  is  $\lambda$ -stable. Then*

$\Delta_\lambda := \{\alpha < \lambda^+ : cf(\alpha) = \alpha \text{ and the } (\lambda, \alpha)\text{-limit model is isomorphic to the } (\lambda, \lambda)\text{-limit model}\}$   
is an end segment of regular cardinals.

We finish this section by presenting a similar result to Theorem 4.22, but without the assumption that  $T$  is closed under direct sums.

**Theorem 4.30.** *Assume  $\mathbf{K}^T$  satisfies Hypothesis 4.1. The following are equivalent.*

- (1)  $\mathbf{K}^T$  is superstable.
- (2) For every  $\lambda \geq 2^{|T|}$ ,  $\mathbf{K}^T$  has uniqueness of limit models of cardinality  $\lambda$ .
- (3) For every  $\lambda \geq 2^{|T|}$ ,  $\mathbf{K}^T$  has a superlimit of cardinality  $\lambda$ .
- (4) For every  $\lambda \geq 2^{|T|}$ ,  $\mathbf{K}^T$  is  $\lambda$ -stable.
- (5) For every  $\lambda \geq 2^{|T|}$ , an increasing chain of  $\lambda$ -saturated models in  $\mathbf{K}^T$  is  $\lambda$ -saturated in  $\mathbf{K}^T$ .

*Proof sketch.*  $\boxed{(1) \rightarrow (4)}$  By (1) and Lemma 4.11 ( $Mod(\tilde{T}), \preceq$ ) has uniqueness of limit models in a tail of cardinals. Then by Fact 2.11  $\tilde{T}$  is stable in a tail of cardinals. Since  $\tilde{T}$  is a first-order theory, the tail has to begin at most in  $2^{|T|}$ .

$\boxed{(4) \rightarrow (2)}$  By Lemma 4.9  $\tilde{T}$  is superstable, then by [Sh78] (see [GrVas17, 1.1.(3)])  $\kappa(\tilde{T}) = \aleph_0$ . The result follows from Fact 2.11 and Lemma 4.27.

$\boxed{(2) \rightarrow (1)}$  Clear.

$\boxed{(4) \rightarrow (5)}$  Similar to (5)  $\rightarrow$  (6) of Theorem 4.22.

$\boxed{(5) \rightarrow (4)}$  The idea is to prove that  $\tilde{T}$  is superstable and then by Lemma 4.9 the result would follow. To prove that  $\tilde{T}$  is superstable, by [Sh78] (see [GrVas17, 1.1.(2)]), it is enough to show that an elementary increasing chain of  $\lambda$ -saturated models in  $\tilde{T}$  is  $\lambda$ -saturated. The proof is similar to (5)  $\rightarrow$  (6) of Theorem 4.25 by using Lemma 4.10.

$\boxed{(2) \rightarrow (3)}$  Similar to (3)  $\rightarrow$  (4) of Theorem 4.22.

$\boxed{(3) \rightarrow (4)}$  Assume for the sake of contradiction that (4) fails, then by Lemma 4.9  $\tilde{T}$  is not superstable. Then by [Sh78] (see [GrVas17, 1.1.(3)])  $\kappa(\tilde{T}) > \aleph_0$ . Let  $\lambda = (2^{|T|})^+$ . Observe that  $\mathbf{K}^T$  is  $\lambda$ -stable since  $\lambda^{|T|} = \lambda$ , so by Fact 2.17.(3) and (3)  $\mathbf{K}^T$  has uniqueness of limit models in  $\lambda$ . Now, by Theorem 4.28.(2), we know that the  $(\lambda, \omega)$ -limit model is not isomorphic to the  $(\lambda, \lambda)$ -limit model, which clearly gives us a contradiction.  $\square$

**Remark 4.31.** *Compared to [GrVas17, 1.3], the above theorem improves the bounds where the nice property shows up from  $\beth_{(2^{|T|})^+}$  to  $2^{|T|}$  in the case of classes of modules satisfying Hypothesis*

4.1. It is worth pointing out that to obtain (3), (5) imply (1) we could have simply quoted [GrVas17, 1.3]. We decided to provide the proofs for those direction to make the paper more transparent and to show the deep connection between  $\mathbf{K}^T$  and  $\tilde{T}$ .

Besides the difference in the bounds between Theorem 4.22 and Theorem 4.30. The techniques are also quite different, while the proof of the first theorem relies more on algebraic notions, the proof of the second theorem relies heavily on model theoretic methods. We do not know if the bounds from Theorem 4.30 can be lowered, but we think it should not be possible.

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