1 Asymptotic notation

1.1 Definitions

Given two functions \( f(x) \) and \( g(x) \), we say that \( f(x) \ll g(x) \) as \( x \to \infty \) if any inequality of the form

\[
|f(x)| \leq 0.000\ldots001 \cdot |g(x)|
\]

eventually holds if you take \( x \) large enough. (How large you need \( x \) to be may depend on the number of zeroes in 0.000…001.) We can also write this as \( g(x) \gg f(x) \) as \( x \to \infty \).

We write \( o(f(x)) \) in an expression as shorthand for “some function \( g(x) \), whose exact form isn’t important, such that \( f(x) \gg g(x) \)”. For example, we have \( \binom{n}{3} = \frac{n^3}{6} + o(n^3) \) as \( n \to \infty \), hiding lower-order terms with \( n^2 \) and \( n \).

We write \( f(x) \sim g(x) \) as \( x \to \infty \) if \( f(x) = g(x) + o(g(x)) \). For example, \( \binom{n}{3} \sim \frac{n^3}{6} \) as \( n \to \infty \). The statement \( f(x) \sim g(x) \) can also be written as \( f(x) = g(x)(1 + o(1)) \), and is equivalent to saying that any inequality of the form

\[
0.999\ldots999 \leq \frac{f(x)}{g(x)} 1.000\ldots001
\]

holds if you take \( x \) is large enough. (Again, how large you need \( x \) to be may depend on the number of zeroes or nines you want in this inequality.)

We can also say “\( f(x) \ll g(x) \)” as \( x \to a \). This is defined in the same way, except that a relationship of the form

\[
|f(x)| \leq 0.000\ldots001 \cdot |g(x)|
\]

must hold not if \( x \) is large enough, but if \( x \) is close enough to \( a \). We define “\( f(x) = o(g(x)) \)” as \( x \to a \) or “\( f(x) \sim g(x) \) as \( x \to a \)” in terms of \( \ll \), in the same way as we did for \( x \to \infty \).

1.2 Practice

1. For what constant \( C \) is \( \binom{n}{3} \sim C n^5 \) true as \( n \to \infty \)?

2. Show that there is some \( N \) such that whenever \( n > N \),

   (a) \( n^2 \) exceeds \( 1000n \).
   (b) \( 2^n \) exceeds \( n^2 \).
   (c) \( 2^n \) exceeds \( n^3 \).
   (d) \( 2^n \) exceeds \( n^{100} \).
   (e) \( 2^n \) exceeds \( 1000 \cdot n^{100} \).
   (f) \( 2^{n^{0.01}} \) exceeds \( n \).
3. Verify as many as you like of the following statements, for $x \to \infty$:

\[
1 \ll \log x \ll (\log x)^{100} \ll x^{0.01} \ll \sqrt{x} \ll x^{2} \ll 2^{x} \ll 3^{x} \ll x! \ll x^{x}.
\]

4. The $n$th harmonic number $H_n$ is equal to the sum $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$.

(a) Given $H_n = \log n + \gamma + o(1)$ as $n \to \infty$, write down an expression for $(H_n)^2$ with an $o(\log n)$ error term.

(b) Given $H_n = \log n + \gamma + \frac{1}{2n} + o\left(\frac{1}{n}\right)$ as $n \to \infty$, write down an expression for $(H_n)^2$ with the best error term you can.

(c) Write down an expression for $H_{2n} - H_n$ with the best error term you can.

5. Prove that if $a_n \sim b_n$ as $n \to \infty$, then

\[
\sum_{n=1}^{\infty} a_n \text{ converges } \iff \sum_{n=1}^{\infty} b_n \text{ converges.}
\]

2 Derivatives

2.1 Definition

For a fixed value $x$ (and function $f$), if there is a constant $C$ such that, as $h \to 0$,

\[f(x + h) = f(x) + Ch + o(h),\]

then we say that $C$ is the derivative of $f$ at $x$, denoted $C = f'(x)$. Equivalent ways to say this using other notation:

\[f(x + h) - f(x) \sim f'(x) \cdot h \quad f(x + h) - (f(x) + f'(x) \cdot h) \ll h\]

In other words, near a point $a$, we can approximate $f(x)$ by $f(a) + f'(a)(x - a)$.

2.2 Practice

1. Prove the product rule: if $h(x) = f(x) \cdot g(x)$, then $h'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$.

2. Prove the chain rule: if $h(x) = f(g(x))$, then $h'(x) = f'(g(x)) \cdot g'(x)$.

3. Prove that, if $f'(x) \neq 0$, then there is a value $y$ (near $x$) such that $f(y) > f(x)$.

(The converse of this statement is that if you want to find the largest value of $f(x)$, you figure out when $f'(x) = 0$.)

4. Prove that if $f(x) = x^n$ for an integer $n > 0$, then $f'(x) = nx^{n-1}$.

5. Using the fact that $e^x \sim x + 1$ as $x \to 0$, prove that if $f(x) = e^x$, then $f'(x) = e^x$.

6. Prove that $f(x) = |x|$ does not have a derivative at 0.
3 More Problems

1. Without finding a formula for \(1^3 + 2^3 + \cdots + n^3\), prove that there is a constant \(C\) such that
\[
1^3 + 2^3 + \cdots + n^3 \sim Cn^4
\]
as \(n \to \infty\). What can you say about \(C\)?

2. Find a function \(f(x)\) such that as \(x \to \infty\), both of the following hold:
   - \(f(x) \gg x^n\) for any integer \(n\);
   - \(f(x) \ll a^x\) for any real number \(a > 1\).

3. (a) Use the inequality \(\sin x \leq x \leq \tan x\) (ask C.J. if you want a proof) to show that \(\sin x \sim x\) as \(x \to 0\).
   (b) Use the statement above to prove that if \(f(x) = \sin x\) then \(f'(x) = \cos x\).

4. (a) Prove that as \(n \to \infty\), \(\log n! \sim n \log n\).
   (b) Prove that if \(n = k!\), then as \(n \to \infty\), \(k \sim \frac{\log n}{\log \log n}\).
   (c) A more precise estimate of \(n!\) is given by Stirling’s formula:
   \[n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n\] as \(n \to \infty\).
   Use this to prove that \(\binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi n}}\) as \(n \to \infty\).

5. The Chebyshev function \(\vartheta(x)\) is defined as
\[
\vartheta(x) = \sum_{p \leq x} \log p
\]
where the sum ranges over only prime \(p\) between 2 and \(x\). The prime-counting function \(\pi(x)\)
is just the number of primes less than or equal to \(x\).
Assuming that \(\vartheta(x) \sim x\) as \(x \to \infty\), prove that \(\pi(x) \sim \frac{x}{\log x}\) as \(x \to \infty\).

6. Prove that the sum \(\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots\) of the reciprocals of the primes diverges.

7. Let \(P_n\) be the set of all \(n\)-digit palindromes. For example:
\[
P_1 = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}
\]
\[
P_2 = \{11, 22, 33, 44, 55, 66, 77, 88, 99\}
\]
\[
P_3 = \{101, 111, 121, \ldots, 979, 989, 999\}
\]
Find the best asymptotic estimate that you can of the sum
\[
S_n = \sum_{p \in P_n} \frac{1}{p}
\]
as \(n \to \infty\).