Number Theory

Theory of Divisors

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ARML Practice 9/29/2013
HMMT 2008/2. Find the smallest positive integer $n$ such that $107n$ has the same last two digits as $n$.

IMO 2002/4. Let $n$ be an integer greater than 1. The positive divisors of $n$ are $d_1, d_2, \ldots, d_k$, where

$$1 = d_1 < d_2 < \cdots < d_k = n.$$ 

Define $D = d_1d_2 + d_2d_3 + \cdots + d_{k-1}d_k$.

(a) Prove that $D < n^2$.

(b) Determine all $n$ for which $D$ is a divisor of $n^2$. 
Two numbers have the same last two digits just when they are the same mod 100, and

\[ n \equiv 107n \pmod{100} \iff n \equiv 7n \pmod{100} \]
\[ \iff 6n \equiv 0 \pmod{100} \]
\[ \iff 6n = 100k \text{ for some } k \]
\[ \iff n = 50 \cdot \frac{k}{3}. \]

So \( n \) must be a multiple of 50, and the smallest such positive number is 50 itself.

2. The IMO problem is left as an exercise.
We can arrange the divisors of 10000 in a square grid:

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Questions:

- How many divisors of 10000 are divisors of 200?
- What is the sum of all the divisors of 10000? (Try to figure out how to avoid using brute force.)
- How many divisors does 10^{100} have?
- How many divisors does 3600 have?
Divisors of 10000

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- Questions:
  - How many divisors of 10000 are divisors of 200?
  - What is the sum of all the divisors of 10000? (Try to figure out how to avoid using brute force.)
  - How many divisors does $10^{100}$ have?
  - How many divisors does 3600 have?
AIME 1998/5. If a random divisor of $10^{99}$ is chosen, what is the probability that it is a multiple of $10^{88}$?

PUMaC 2011/NT A1. The only prime factors of an integer $n$ are 2 and 3. If the sum of the divisors of $n$ (including $n$ itself) is 1815, find $n$.

Original. How many divisors $x$ of $10^{100}$ have the property that the number of divisors of $x$ is also a divisor of $10^{100}$?
AIME 1998/5. The divisors of $10^{99}$ form a $100 \times 100$ grid. In the grid, the multiples of $10^{88}$ are the numbers below and to the right of $10^{88}$, which form a $12 \times 12$ grid. So the probability is

\[ \frac{12 \cdot 12}{100 \cdot 100} = 0.0144. \]
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PUMaC 2011/NT A1. First note that $1815$ factors as $3 \cdot 5 \cdot 11^2$. If $n = 2^a \cdot 3^b$, the sum of its divisors is

$$(1 + 2 + 4 + \cdots + 2^a)(1 + 3 + 9 + \cdots + 3^b).$$

The sums of powers of 2 begin $1, 3, 7, 15, 31, \ldots$ and the sums of powers of 3 begin $1, 4, 13, 40, 121, \ldots$. At this point we spot that $15 \cdot 121 = 1815$. This is $1 + 2 + 4 + 8$ times $1 + 3 + 9 + 27 + 81$, so $n$ is $8 \cdot 81 = 648$. 

Original. Since $10^{100} = 2^{100} \cdot 5^{100}$, $x$ must also be of the form $2^a \cdot 5^b$, where $0 \leq a \leq 100$ and $0 \leq b \leq 100$.

The divisors of $x$ form their own grid, with $a + 1$ columns (there are $a + 1$ choices for the power of 2, namely $2^0, 2^1, 2^2, \ldots, 2^a$) and $b + 1$ rows (there are $b + 1$ choices for the power of 5). The total number of divisors of $x$ is $(a + 1)(b + 1)$.

If this number is also a divisor of $10^{100}$, then both $a + 1$ and $b + 1$ must be products of 2’s and 5’s. There are no further restrictions on $x$. So $a + 1$ and $b + 1$ can each be one of:

$$1, 2, 4, 8, 16, 32, 64, \quad 5, 10, 20, 40, 80, \quad 25, 50, 100.$$  

There are 15 possibilities for $a$ and for $b$, so there are $15^2 = 225$ possibilities for $x$. 
Warm-up Basics of divisors Taking equations mod $n$

Taking equations mod $n$
Pythagorean triples

**Problem**

*If $x, y, z$ are integers and $x^2 + y^2 = z^2$, show that 60 divides $xyz$.***
Taking equations mod $n$

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- All three of $x, y, z$ cannot be odd, since odd + odd = even. So $xyz$ is even.
Taking equations mod \( n \)

Pythagorean triples

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- All three of \( x, y, z \) cannot be odd, since odd + odd = even. So \( xyz \) is even.

- Since \( 1^2 \equiv 2^2 \equiv 1 \) (mod 3), all perfect squares are 0 or 1 mod 3. But \( x^2 + y^2 \equiv z^2 \) (mod 3) is not solved by making each of \( x^2, y^2, \) and \( z^2 \) be 1 mod 3. So one is 0 mod 3, and so \( xyz \) is divisible by 3.
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- Mod 5, we have $1^2 \equiv 4^2 \equiv 1$ and $2^2 \equiv 3^2 \equiv -1$. So $x^2 + y^2 \equiv z^2 \pmod{5}$ can look like $0 \pm 1 \equiv \pm 1$ or $1 - 1 \equiv 0$. So one of $x, y, z$ is 0 mod 5, and $xyz$ is divisible by 5.

These mean $xyz$ is divisible by 30. Getting 60 is left as an exercise (Hint: try mod 8.)
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Taking equations mod $n$

Competition-level problems

**Original.** If $x, y, z$ are integers and $x^2 + y^2 = 3z^2$, show that $x = y = z = 0$.

**PUMaC 2007/NT B2.** How many positive integers $n$ are there such that $n + 2$ divides $(n + 18)^2$?

**British MO 2005/6.** Let $n$ be an integer greater than 6. Prove that if $n - 1$ and $n + 1$ are both prime, then $n^2(n^2 + 16)$ is divisible by 720.

**PUMaC 2009/NT A3.** Find all prime numbers $p$ which can be written as $p = a^4 + b^4 + c^4 - 3$ for some primes (not necessarily distinct) $a$, $b$, and $c$. 
Taking equations mod $n$

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Taking equations mod $n$

Solutions

**Original.** If $x^2 + y^2 = 3z^2$, then $x^2 + y^2 \equiv 0 \pmod{3}$, which is only possible if $x \equiv y \equiv 0 \pmod{3}$. So both $x$ and $y$ are divisible by 3, so $x^2 + y^2$ is divisible by 9, and therefore $z^2$ is divisible by 3.

We now have $(x/3)^2 + (y/3)^2 = 3(z/3)^2$, so the same is true of $x/3, y/3, z/3$. But the numbers cannot have infinitely many factors of 3 unless they are all 0.
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**PUMaC 2007/NT B2.** Since $n + 18 \equiv 16 \pmod{n + 2}$, $(n + 18)^2 \equiv 16^2 \pmod{n + 2}$ We are given $(n + 18)^2 \equiv 0 \pmod{n + 2}$, so $16^2 \equiv 0 \pmod{n + 2}$, which means $n + 2$ divides 256. Therefore $n + 2$ is one of $2^2, 2^3, \ldots, 2^8$, which gives 7 solutions.
BMO 2005/6. Divisibility by 144 is easy. Neither \( n + 1 \) nor \( n - 1 \) is even, so \( n \) must be even; and neither \( n + 1 \) nor \( n - 1 \) is divisible by 3, so \( n \) must be divisible by 3. Therefore \( n = 6k \), and

\[
    n^2(n^2 + 16) = (6k)^2((6k)^2 + 16) = 144 \cdot k^2(9k^2 + 4).
\]

Now all we need is divisibility by 5. Since neither \( n + 1 \) nor \( n - 1 \) is divisible by 5, we have one of \( n \equiv 0, 2, 3 \pmod{5} \). Fortunately,

\[
\begin{align*}
0^2(0^2 + 16) &= 0 \equiv 0 \quad \pmod{5} \\
2^2(2^2 + 16) &= 80 \equiv 0 \quad \pmod{5} \\
3^2(3^2 + 16) &= 225 \equiv 0 \quad \pmod{5}.
\end{align*}
\]

So in all three cases, \( n^2(n^2 + 16) \) is divisible by 5.
Taking equations mod \( n \)

Solutions

PUMaC 2009/NT A3. The primes 2, 3, and 5 have the following property: if \( p \) is one of 2, 3, or 5, then either \( a \equiv 0 \pmod{p} \) or \( a^4 \equiv 1 \pmod{p} \). This is easy to check:

\[
\begin{align*}
1^4 & \equiv 1 & \pmod{2} \\
1^4 & \equiv 2^4 \equiv 1 & \pmod{3} \\
1^4 & \equiv 2^4 \equiv 3^4 \equiv 4^4 \equiv 1 & \pmod{5}.
\end{align*}
\]

Suppose none of \( a, b, \) or \( c \) are 2. They are prime, so not divisible by 2. But then

\[
p = a^4 + b^4 + c^4 - 3 \equiv 1 + 1 + 1 - 3 \equiv 0 \pmod{2}
\]

and \( p \) is divisible by 2 (but it’s easy to check \( p = 2 \) doesn’t work). So one of \( a, b, \) or \( c \) has to be 2.

The same argument shows that one of \( a, b, \) or \( c \) has to be 3, and one has to be 5. This means \( p = 2^4 + 3^4 + 5^4 - 3 = 719 \).