Graph theory

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1 Basic results

We begin by collecting some basic facts which can be proved via "bare-hands" techniques.

1. The sum of all of the degrees is equal to twice the number of edges. Deduce that the number of odd-degree vertices is always an even number.

   Solution: By counting in two ways, we see that the sum of all degrees equals twice the number of edges.

2. A graph is called *bipartite* if it is possible to separate the vertices into two groups, such that all of the graph's edges only cross between the groups (no edge has both endpoints in the same group). Prove that this property holds if and only if the graph has no cycles of odd length.

   Solution: Separate into connected components. For each, choose a special vertex, and color based on parity of length of shortest path from that special vertex.

3. Every connected graph with all degrees even has an *Eulerian circuit*, i.e., a walk that traverses each edge exactly once.

   Solution: Start walking from a vertex v_1 without repeating any edges, and observe that by the parity condition, the walk can only get stuck at v_1, so we get one cycle. If we still have more edges left to hit, connectivity implies that some vertex v_2 on our current walk is adjacent to an unused edge, so start the process again from v_2. Splice the two walks together at v_2, and repeat until done.

4. Suppose that a graph has at least as many edges as vertices. Show that it contains a cycle.

   Solution: As long as there are vertices with degree exactly 1, delete both the vertex and its incident edge. Also delete all isolated vertices. These operations preserve $E \geq V$, but we can never delete everything because once $V = 1$, E must be 0, so we can never get down to only 1 vertex or less.

   Therefore we end up with a nonempty graph with all degrees ≥ 2, and by taking a walk around and eventually hitting itself, we get a cycle.

5. Suppose that the graph G has all degrees at most Δ. Prove that it is possible to color the vertices of G using $\leq \Delta + 1$ colors, such that no pair of adjacent vertices receives the same color.

   Solution: Consider the greedy algorithm for coloring vertices.

6. Let G_1, G_2, G_3 be three (possibly overlapping) graphs on the same vertex set, and suppose that G_1 can be properly colored with 2 colors, G_2 can be properly colored with 3 colors, and G_3 can be properly colored with 4 colors. Let G be the graph on the same vertex set, formed by taking the union of the edges appearing in G_1, G_2, G_3. Prove that G can be properly colored with 24 colors.

   Solution: Product coloring.

2 Matching

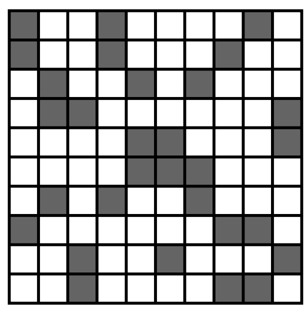
Consider a bipartite graph $G = (V, E)$ with partition $V = A \cup B$. A *matching* is a collection of edges which have no endpoints in common. We say that *A has a perfect matching to B* if there is a matching which hits every vertex in A.

Theorem. *(Hall's Marriage Theorem) For any set $S \subset A$, let $N(S)$ denote the set of vertices (necessarily in B) which are adjacent to at least one vertex in S. Then, A has a perfect matching to B if and only if $|N(S)| \geq |S|$ for **every** $S \subset A$.*

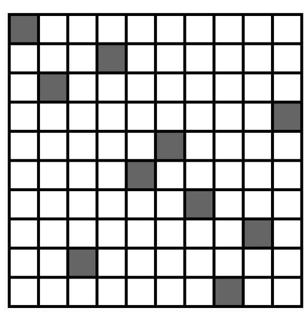
This has traditionally been called the "marriage" theorem because of the possible interpretation of edges as "acceptable" pairings, with the objective of maximizing the number of pairings. In real life, however, perhaps there may be varying degrees of "acceptability." This may be formalized by giving each vertex (in both parts) an ordering of its incident edges. Then, a matching M is called *unstable* if there is an edge $e = ab \notin M$ for which both a and b both prefer the edge e to their current partner (according to M).

Theorem. *(Stable Marriage Theorem) A stable matching always exists, for every bipartite graph and every collection of preference orderings.*

1. You are given a 10×10 grid, with the property that in every row, exactly 3 squares are shaded, and in every column, exactly 3 squares are shaded. An example is below.



Prove that there must always be a shaded *transversal*, i.e., a choice of 10 shaded squares such that no two selected squares are in the same row or column. An example is below.



Solution: Hall's theorem.

2. (Hall with deficiency.) In a bipartite graph, every subset S of the left side has $|N(S)| \geq |S| - 1$. Prove that there is an almost-perfect matching, in the sense that there is a matching which involves all but at most one vertex of the left side.

3. (Multi-Hall.) In a bipartite graph, every subset S of the left side has $|N(S)| \geq 2|S|$. Prove that there is a perfect 1-to-2 matching, in the sense that each vertex of the left is matched to a pair of vertices on the right, and all of the pairs on the right are disjoint.

4. Every k-regular bipartite graph contains a perfect matching.

5. (Diestel 2.11.) Suppose that a bipartite graph has bipartition $A \cup B$, and for every edge ab with $a \in A$ and $b \in B$, we have $\deg(a) \geq \deg(b)$. Prove that there is a perfect matching from A to B.

3 Planarity

When we represent graphs by drawing them in the plane, we draw edges as curves, permitting intersections. If a graph has the property that it can be drawn in the plane without any intersecting edges, then it is called *planar*. Here is the tip of the iceberg. One of the most famous results on planar graphs is the Four-Color Theorem, which says that every planar graph can be properly colored using only four colors. But perhaps the most useful planarity theorem in Olympiad problems is the Euler Formula:

Theorem. *Every connected planar graph satisfies $V - E + F = 2$, where V is the number of vertices, E is the number of edges, and F is the number of faces.*

Solution: Actually prove that $V - E + F = 1 + C$, where C is the number of connected components. Each connecting curve is piecewise-linear, and if we add vertices at the corners, this will keep $V - E$ invariant. Now we have a planar graph where all connecting curves are straight line segments.

Then induction on $E + V$. True when $E = 0$, because $F = 1$ and $V = C$. If there is a leaf (vertex of degree 1), delete both the vertex and its single incident edge, and $V - E$ remains invariant. If there are no leaves, then every edge is part of a cycle. Delete an arbitrary edge, and that will drop E by 1, but also drop F by 1 because the edge was part of a cycle boundary, and now that has merged two previously distinct faces.

Now, use the theorem to solve the following problems:

1. Prove that K_5 is not planar.

2. Prove that $K_{3,3}$ is not planar.

3. Prove that $K_{4,4}$ is not planar.

4. Prove that every planar graph can be properly colored using at most 6 colors.

The Euler criterion immediately implies that every connected graph has at least $E - (3V - 6)$ crossings. As it turns out, one can do much better:

Theorem. *(Ajtai, Chvátal, Newborn, Szemeredi; Leighton.) Every connected graph with $E \geq 4V$ has at least $\frac{E^3}{64V^2}$ crossings.*

4 Ramsey theory

Complete disorder is impossible.

— T. S. Motzkin, on the theme of Ramsey Theory.

The *Ramsey Number* $R(s,t)$ is the minimum integer n for which **every** red-blue coloring of the edges of K_n contains a completely red K_s or a completely blue K_t. Ramsey's Theorem states that $R(s,t)$ is always finite, and we will prove this in the first exercise below. The interesting question in this field is to find upper and lower bounds for these numbers, as well as for quantities defined in a similar spirit.

1. Prove by induction that $R(s,t) \leq \binom{s+t-2}{s-1}$. Note that in particular, $R(3,3) \leq 6$.

Solution: Observe that $R(s,t) \le R(s-1,t)+R(s,t-1)$, because if we have that many vertices, then if we select one vertex, then it cannot simultaneously have $< R(s-1,t)$ red neighbors and $< R(s,t-1)$ blue neighbors, so we can inductively build either a red K_s or a blue K_t. But

$$\binom{(s-1)+t-2}{(s-2)} + \binom{s+(t-1)-2}{s-1} = \binom{s+t-2}{s-1},$$

because in Pascal's Triangle the sum of two adjacent guys in a row equals the guy directly below them in the next row.

2. Show that $R(t,t) \le 2^{2t}$. Then show that $R(t,t) > 2^{t/2}$ for $t \ge 3$, i.e., there is a red-blue coloring of the edges of the complete graph on $2^{t/2}$, such that there are no monochromatic K_t.

   Solution: The first bound follows immediately from the Erdős-Szekeres bound. The second is an application of the probabilistic method. Let $n = 2^{t/2}$, and consider a random coloring of the edges of K_n, where each edge independently receives its color with equal probabilities. For each set S of t vertices, define the event E_S to be when all $\binom{t}{2}$ edges in S are the same color. It suffices to show that $\mathbb{P}[\text{some } E_S \text{ occurs}] < 1$. But by the union bound, the LHS is

$$\begin{aligned}\binom{n}{t} \cdot \left(2 \cdot 2^{-\binom{t}{2}}\right) &\le \frac{n^t}{t!} \cdot 2 \cdot 2^{-\frac{t^2}{2}+\frac{t}{2}} \\ &= \frac{\left(2^{t/2}\right)^t}{t!} \cdot 2 \cdot 2^{-\frac{t^2}{2}+\frac{t}{2}} \\ &= 2 \cdot \frac{2^{t/2}}{t!}.\end{aligned}$$

   This final quantity is less than 1 for all $t \ge 3$.

3. (IMO 1964/4.) Seventeen people correspond by mail with one another—each one with all the rest. In their letters only 3 different topics are discussed. Each pair of correspondents deals with only one of these topics. Prove that there are at least 3 people who write to each other about the same topic.

   Solution: This is asking us to prove that the 3-color Ramsey Number $R(3,3,3)$ is ≤ 17. By the same observation as in the previous problem, $R(a,b,c) \le R(a-1,b,c)+R(a,b-1,c)+R(a,b,c-1)-1$. Then using symmetry, $R(3,3,3) \le 3R(3,3,2)-1$. It suffices to show that $R(3,3,2) \le 6$. But this is immediate, because if we have 6 vertices, if we even use the 3rd color on a single edge, we already get a K_2. So we cannot use the 3rd color. But then from above, we know $R(3,3) \le 6$, so we are done.

5 Blue

1. (Putnam 1957/A5.) Let S be a set of n points in the plane such that the greatest distance between two points of S is 1. Show that at most n pairs of points of S are at distance 1 apart.

   Solution: Show that if there is any vertex of degree ≥ 3 in the unit distance graph, then it has a neighbor of degree 1 in the unit distance graph. Pulling off that neighbor by induction solves the problem, or else all degrees are ≤ 2, at which point the edge bound follows.

2. Every tournament (complete graph with each edge oriented in some direction) contains a Hamiltonian directed path (hitting every vertex exactly once).

3. (Romania 2006.) Each edge of a polyhedron is oriented with an arrow such that every vertex has at least one edge directed toward it, and at least one edge directed away from it. Show that some face of the polyhedron has its boundary edges coherently oriented in a circular direction.

   Solution: A directed cycle in the graph exists by simply following out-edges until we repeat vertices. If it has stuff inside it, then one can cut the cycle with a directed path, and then there is a shorter directed cycle. Compare the sizes of cycles by the number of faces they contain.

4. (Monotone paths.) Show that for every even n, it is possible to label the edges of K_n with the distinct integers $1, 2, \ldots, \binom{n}{2}$, in such a way that no increasing walk contains more than $n-1$ edges. An increasing walk is a sequence of vertices $v_0, v_1, \ldots, v_t$ such that the labels of the edges $v_i v_{i+1}$ increase with i. The vertices $v_0, \ldots, v_t$ are not required to be distinct—that is the difference between the definitions of walks and paths.

5. (Sweden 2010.) A town has $3n$ citizens. Any two persons in the town have at least one common friend in this same town. Show that one can choose a group consisting of n citizens such that every person of the remaining $2n$ citizens has at least one friend in this group of n.

   Solution: The codegree condition implies that the diameter of the graph is at most 2. We prove that every n-vertex graph with diameter ≤ 2 has a dominating set (a subset S of vertices such that every other vertex is either in, or has a neighbor in S) of size only $\leq \sqrt{n \log n} + 1$. To see this, let $p = \sqrt{\frac{\log n}{n}}$.

   Observe that since the diameter is at most 2, if any vertex has degree $\leq np$, then its neighborhood already is a dominating set of suitable size. Therefore, we may assume that all vertices have degree strictly greater than np. It feels "easy" to find a small dominating set in this graph because all degrees are high. Consider a random sample of np vertices (selected uniformly at random, with replacement), and let S be their union. Note that $|S| \leq np$. Now the probability that a particular fixed vertex v fails to have a neighbor in S is strictly less than $(1-p)^{np}$, because we need each of np independent samples to miss the neighborhood of v. This is at most $e^{-np^2} \leq e^{-\log n} = n^{-1}$. Therefore, a union bound over the n choices of v produces the result.

6. (Bondy 1.5.9.) There are n points in the plane such that every pair of points has distance ≥ 1. Show that there are at most $3n$ (unordered) pairs of points that span distance exactly 1 each.

   Solution: The unit distance graph is planar.

7. (Prüfer.) A graph with vertex set $\{1, \ldots, n\}$ is a *spanning tree* if it is a tree which includes all of those n vertices. It turns out that there is a surprisingly beautiful formula for the number of spanning trees on $\{1, \ldots, n\}$: it is just n^{n-2}.

8. Let n be an even integer. It is possible to partition the edges of K_n into exactly $n-1$ perfect matchings. (In this context of non-bipartite graphs, a perfect matching is a collection of $n/2$ edges that touch every vertex exactly once.) We can interpret that as a way to run a round-robin sports tournament among n teams: on each of $n-1$ days, the n teams pair up according to the day's perfect matching, and each of the $n/2$ edges tells who plays who that day. There are $n/2$ simultaneous games on each of the $n-1$ days.

   On each of the $n-1$ days, there are $n/2$ winning teams from the $n/2$ games. So, there are $n/2$ winners of Day 1, $n/2$ winners of Day 2, ..., and $n/2$ winners of Day $(n-1)$. Prove that no matter how the $\binom{n}{2}$ individual games turned out, it is always possible (after all of the games) to select one team who was a winner of Day 1, one team who was a winner of Day 2, ..., and one team who was a winner of Day $(n-1)$, such that we don't pick the same team twice. Note that since there are n teams in total, this selection will always leave exactly one team out.