

# Graph Theory: Introduction

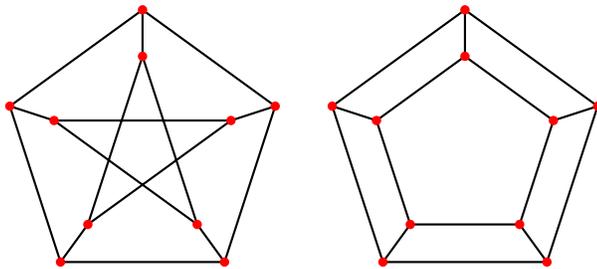
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At first, graph theory may seem to be an *ad hoc* subject, and in fact the elementary results have proofs of that nature. The methods recur, however, and the way to learn them is to work on problems. Later, when you see an Olympiad graph theory problem, hopefully you will be sufficiently familiar with graph-theoretic arguments that you can rely on your own ingenuity to invent an approach.

## 1 Warm-up

1. Are these two graphs the same?



**Solution:** No. The Petersen graph (to the left) has no cycles of length 4.

2. There are 21 friends at a party. Each person counts how many of the other people are he/she has met before. Is it possible for the sum of these counts to be an odd number?
3. Show that every graph has at least two vertices with equal degree.

**Solution:** Pigeonhole: all degrees between 0 and  $n - 1$ , but if we have a 0, we cannot have an  $n - 1$ . So there are  $n - 1$  available degrees and  $n$  vertices.

4. Let  $G$  be self-complementary (replacing edges with non-edges, and vice versa, gives back the same graph). Prove that the number of vertices must be 0 or 1 modulo 4.

**Solution:** Sum  $S$  of degrees is even. The degree of  $v$  plus its degree in the complement is exactly  $n - 1$ . Self complementary, so summing this we get  $2S = n(n - 1)$ . The result follows since  $S$  is even.

## 2 Tools

### 2.1 Bare-hands

Each of these well-known results can be proved without any fancy theorems. **Prove them.**

1. The sum of all of the degrees is equal to twice the number of edges. Deduce that the number of odd-degree vertices is always an even number.

**Solution:** By counting in two ways, we see that the sum of all degrees equals twice the number of edges.

2. A graph is called *bipartite* if it is possible to separate the vertices into two groups, such that all of the graph's edges only cross between the groups (no edge has both endpoints in the same group). Prove that this property holds if and only if the graph has no cycles of odd length.

**Solution:** Separate into connected components. For each, choose a special vertex, and color based on parity of length of shortest path from that special vertex.

3. Every connected graph with all degrees even has an *Eulerian circuit*, i.e., a walk that traverses each edge exactly once.

**Solution:** Start walking from a vertex  $v_1$  without repeating any edges, and observe that by the parity condition, the walk can only get stuck at  $v_1$ , so we get one cycle. If we still have more edges left to hit, connectivity implies that some vertex  $v_2$  on our current walk is adjacent to an unused edge, so start the process again from  $v_2$ . Splice the two walks together at  $v_2$ , and repeat until done.

4. Suppose that a graph has at least as many edges as vertices. Show that it contains a cycle.

**Solution:** As long as there are vertices with degree exactly 1, delete both the vertex and its incident edge. Also delete all isolated vertices. These operations preserve  $E \geq V$ , but we can never delete everything because once  $V = 1$ ,  $E$  must be 0, so we can never get down to only 1 vertex or less.

Therefore we end up with a nonempty graph with all degrees  $\geq 2$ , and by taking a walk around and eventually hitting itself, we get a cycle.

5. Suppose that the graph  $G$  has all degrees at most  $\Delta$ . Prove that it is possible to color the vertices of  $G$  using  $\leq \Delta + 1$  colors, such that no pair of adjacent vertices receives the same color.

**Solution:** Consider the greedy algorithm for coloring vertices.

6. Let  $G$  be a graph. It is possible to partition the vertices into two groups such that for each vertex, at least half of its neighbors ended up in the other group.

**Solution:** Take a max-cut: the bipartition which maximizes the number of crossing edges.

7. Let  $\delta$  be the minimum degree of  $G$ , and suppose that  $\delta \geq 2$ . Then  $G$  contains a cycle of size  $\geq \delta + 1$ . In particular, it contains a path with  $\geq \delta$  edges.

**Solution:** Take a longest path. Let  $v$  be its last endpoint. By maximality, every one of  $v$ 's  $\geq \delta$  neighbors lie on the path. So path has length  $\geq \delta + 1$ .

8. (Dirac's Theorem.) Let  $G$  be a graph on  $n$  vertices with all degrees at least  $n/2$ . Show that  $G$  has a Hamiltonian cycle.

**Solution:** Suppose the longest path has  $t$  vertices  $x_1, \dots, x_t$ . We will show there is a cycle of  $t$  vertices as well. Suppose not. All neighbors of  $x_1$  and  $x_t$  must lie on the path or else it is not longest. Minimum degree condition implies that both have degree  $\geq t/2$ . But if  $x_1 \sim x_k$ , then  $x_t \not\sim x_{k-1}$  or else we can re-route to get a cycle. So, each of  $x_1$ 's  $t/2$  neighbors on the path prohibit a potential neighbor of  $x_t$ . Yet  $x_t$ 's neighbors come from indices  $1 \dots t - 1$ , so there is not enough space for  $x_t$  to have  $t/2$  neighbors there, avoiding the prohibited ones.

Now if this longest path is not the full  $n$  vertices, then we get a cycle  $C$  missing some vertex  $x$ . But min-degree  $n/2$  implies that the graph is connected (smallest connected component is  $n/2 + 1$ ), so there is a shortest path from  $x$  to  $C$ , and adding this to the cycle gives a longer path than  $t$ , contradiction.

9. (Diestel 1.7) If  $G$  is also known to be connected,  $G$  actually contains a path with  $\geq \min\{2\delta, V - 1\}$  edges.

**Solution:** The same solution as above for Dirac's Theorem works here.

10. Consider a graph where every vertex has degree exactly  $2k$ . Show that it is possible to orient each edge such that the maximum in-degree is exactly  $k$ .

**Solution:** Direct along an Eulerian circuit.

### 3 Olympiad problems

1. (Tournament of the towns 1986.) 20 football teams take part in a tournament. On the first day all the teams play one match. On the second day all the teams play a further match. Prove that after the second day it is possible to select 10 teams, so that no two of them have yet played each other.

**Solution:** This is a graph on 20 vertices whose edge set is a union of 2 matchings. So all degrees  $\leq 2$ , and graph is a disjoint union of even cycles. Clearly possible to get independent set of size half.

2. (BAMO 2006/1.) All the chairs in a classroom are arranged in a square  $n \times n$  array (in other words,  $n$  columns and  $n$  rows), and every chair is occupied by a student. The teacher decides to rearrange the students according to the following two rules:

- Every student must move to a new chair.
- A student can only move to an adjacent chair in the same row or to an adjacent chair in the same column. In other words, each student can move only one chair horizontally or vertically.

(Note that the rules above allow two students in adjacent chairs to exchange places.) Show that this procedure can be done if  $n$  is even, and cannot be done if  $n$  is odd.

**Solution:** For even  $n$ , swap pairs of adjacent students. Now consider odd  $n$ . Start at an arbitrary student, and see where he goes. Draw an arrow from the vertex corresponding to his original seat to the vertex corresponding to his destination. But his destination must have been vacated, so see where that person goes. Draw another arrow corresponding to that move. Keep going, building a directed path. Note that since we can never put two people into the same seat, this directed path will never intersect itself, until it makes an arrow back into its initial vertex. So, we have just made a directed cycle.

Delete all vertices of this cycle, and continue; after we have exhausted all of the students, we should have a decomposition into a disjoint union of directed cycles. The key observation is that every cycle has even length. Indeed, for each cycle, the number of right-pointing arrows equals the number of left-pointing arrows, because each move is of distance exactly 1 in either direction. Similarly, the number of upward arrows equals the number of downward arrows. But it is impossible to decompose an odd number of vertices into a disjoint union of even cycles, contradiction.

3. (From *Mathematical Circles*.) In Orientalia all the roads are one-way roads, and you can reach each town from any other by driving along no more than two roads. One of the roads is closed for repair, but it is still possible to drive from each town to any other. Prove that now this can be done by driving along at most three roads.

**Solution:** Consider a vertex  $v$ . Partition the vertices into  $v$ ,  $N^+(v)$ , and its second-out-neighborhood, which actually includes the rest of the graph. Suppose we are trying to reach  $w \in N^{++}(v)$ . Note that there is a path  $vw$  with  $u \in N^+(v)$ .

CASE 1: the deleted edge is  $uw$ . Since it is still possible to reach  $w$  from  $v$ , there is some such path, and suppose the last vertex it visits before  $w$  is  $x$ . If  $x \in N^+(v)$ , then we already have a route  $vxw$ . If

$x \in N^{++}(v)$ , then it is still possible to reach  $x$  from  $v$  in 2 steps since we deleted only one edge, and then 1 further step gets to  $w$ .

CASE 2: the deleted edge is  $vu$ . Since it is still possible to reach  $w$  from  $v$ , some such path starts off with some  $vx$ . Now in the old graph there was a 2-path from  $x$  to  $w$ , and clearly this cannot use the edge  $vu$ , so we can add it to  $vx$  and get a 3-path.

FINAL CASE:  $w \in N^+(v)$ . Just as in Case 2, there is some  $vx$  still intact. The original 2-path from  $x$  to  $w$  cannot use  $vu$  because the edge  $vx$  was already oriented in that direction. So we again can join  $vx$  to the original 2-path from  $x$  to  $w$ .

4. (BAMO 2004/3.) NASA has proposed populating Mars with 2,004 settlements. The only way to get from one settlement to another will be by a connecting tunnel. A bored bureaucrat draws on a map of Mars, randomly placing  $N$  tunnels connecting the settlements in such a way that no two settlements have more than one tunnel connecting them. What is the smallest value of  $N$  that guarantees that, no matter how the tunnels are drawn, it will be possible to travel between any two settlements?

**Solution:** This is asking for the max number of edges in a disconnected graph, after which we add 1. Suppose we have a maximal disconnected graph. Then each connected component must be a clique. By convexity, the answer corresponds to a single  $K_{2003}$  plus an isolated vertex.

5. (St. Petersburg 1996/4) In a group of several people, some are acquainted with each other and some are not. Every evening, one person invites all of his acquaintances to a party and introduces them to each other. Suppose that after each person has arranged at least one party, some two people are still unacquainted. Prove that they will not be introduced at the next party.

**Solution:** We prove the stronger statement that at the end of this process, all connected components are cliques. For this, it suffices to show that if the initial graph was connected, then the final graph is a clique. Use induction: let  $v$  be the last guy, and let  $G'$  be the acquaintance graph right before his party. Then  $G' - v$  is a disjoint union of cliques, and each clique is connected back to  $v$ . So when  $v$  holds his party, everything connects into a big clique.

6. (IMO Shortlist 2004/C3) The following operation is allowed on a finite graph: Choose an arbitrary cycle of length 4 (if there is any), choose an arbitrary edge in that cycle, and delete it from the graph. For a fixed integer  $n \geq 4$ , find the least number of edges of a graph that can be obtained by repeated applications of this operation from a complete graph on  $n$  vertices (where each pair of vertices are joined by an edge).

**Solution:** First we show that we cannot end up with any graph with  $\leq n - 1$  edges. We are only breaking cycles, so we cannot destroy connectivity. Therefore, any final graph with  $\leq n - 1$  edges must have exactly  $n - 1$  edges, and be a tree, hence bipartite. But if we consider the reverse process, observe that if we start from a bipartite graph and complete  $C_4$ 's, we will stay bipartite, and  $K_n$  is not bipartite!

It remains to find an  $n$ -edge graph that we can reach. I think one such graph is a triangle plus a single path leading out of one of the vertices of the triangle. See if you can prove this.

7. (USAMO 1989/2) The 20 members of a local tennis club have scheduled exactly 14 two-person games among themselves, with each member playing in at least one game. Prove that within this schedule there must be a set of 6 games with 12 distinct players.

**Solution:** Take a maximal matching. Suppose it has  $t \leq 5$  edges. Then there are  $20 - 2t$  vertices not in this matching, and since matching is maximal, those vertices span no edges. But every vertex has degree  $\geq 1$ , so we must have  $\geq 20 - 2t$  edges from those leftover vertices. This requires a total of  $\geq (20 - 2t) + t = 20 - t \geq 15$  edges, which is too many.

8. (BAMO 2005/4) There are 1000 cities in the country of Euleria, and some pairs of cities are linked by dirt roads. It is possible to get from any city to any other city by traveling along these roads. Prove

that the government of Euleria may pave some of the roads so that every city will have an odd number of paved roads leading out of it.

**Solution:** The key is that 1000 is even. Reduce to the case when the graph is a spanning tree. (One can do breadth-first-search, for example.) Take a leaf and its parent  $v$ , and pave the edge between them. Delete both vertices, and let the connected components of the remainder be  $C_1, \dots, C_k$ . Note that since we only deleted 2 vertices, the sum of all remaining component sizes is still even, so the number of odd components is even.

We are going to induct into each component, but for this we need all components to be even. For each odd component, add back a copy of  $v$  with an edge in the same place it used to be, so that now it is an even graph (but indeed still smaller than the original graph, by at least 1 vertex). Inductively solve every component, and observe that the artificial  $v$ 's get odd numbers of paved edges an even number of times, since there was an even number of odd components. So still the final  $v$  gets an odd number of paved edges because of the initial paving.

9. (MOP Test 2008/8/1) Prove that if the edges of a complete graph on  $n$  vertices are colored such that no color is assigned to more than  $n - 2$  edges, there must contain a 3-cycle in which each edge is a distinct color.

**Solution:** (By David B. Rush) Like dude ok try for a contradiction. Consider a monochromatic connected component of a color, say, yellow to be a connected component for which there exists a yellow path from each vertex to any other. Now consider the BIGGEST one. If there exists a vertex outside, note that the edges from this vertex to any vertex in the maximal monochromatic connected component must all be the same color, contradicting maximality. Hence the monochromatic thing is the WHOLE GRAPH. Take a monochromatic spanning tree. It has at least  $n-1$  edges. 7/0.9 :)

10. (Russia 1998/48) There are 1998 cities in Russia, each being connected (in both directions) by flights to three other cities. Any city can be reached by any other city by a sequence of flights. The KGB plans to close off 200 cities, no two joined by a single flight. Show that this can be done so that any open city can be reached from any other open city by a sequence of flights only passing through open cities.