21-228 Discrete Mathematics

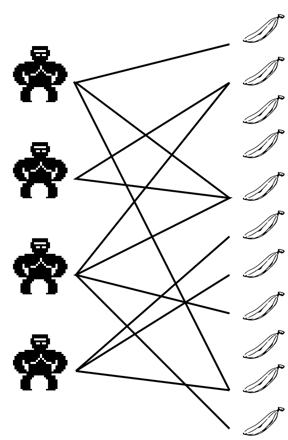
Assignment 8

Due Fri May 3, at start of class

Notes: Collaboration is permitted except in the writing stage. Also, please justify every numerical answer with an explanation.

1. Let G be an arbitrary graph. It will always have spanning subgraphs (containing all of the vertices) which are bipartite. Among all of them, pick one which has the maximum number of edges, and call this subgraph H. This is called a maximum bipartite subgraph of G. The bipartition divides the vertices of H (and G) into two sides, L and R. Prove that every vertex v has the property that (according to G) at least half of its incident edges go across to the other side. For example, if v were in L, and v happened to have degree 10 in G, then at least 5 of those edges would have their other endpoint in R.

2. King Kong and his friends are sharing some bananas. Each gorilla has a particular subset of the bananas that he/she is interested in. (Perhaps some of the others are mushy.) Coincidentally, it turns out that for each set of S gorillas, the collection of bananas that any of them are interested in is of size at least $2|S|$. Prove that there is a way to give two bananas to each gorilla, so that every gorilla receives only bananas that he/she was interested in.



   In graph theoretic language: Let G be a bipartite graph with bipartition $V = A \cup B$. Suppose that for every set $S \subset A$, its combined neighborhood satisfies $|N(S)| \geq 2|S|$. Here, $N(S)$ the collection of all $w \in B$ for which w has a neighbor in S. It is not necessary for every vertex in S to be adjacent to w. Then, prove that G admits a perfect 1-to-2 matching, i.e, a choice of two neighbors for every $v \in A$, with no vertex in B being assigned twice.

3. Let n be an even integer. We showed some time ago in class that it is possible to partition the edges of K_n into exactly $n-1$ perfect matchings. (In this context of non-bipartite graphs, a perfect matching is a collection of $n/2$ edges that touch every vertex exactly once.) We can interpret that as a way to run a round-robin sports tournament among n teams: on each of $n-1$ days, the n teams pair up according to the day's perfect matching, and each of the $n/2$

edges tells who plays who that day. There are $n/2$ simultaneous games on each of the $n-1$ days.

On each of the $n-1$ days, there are $n/2$ winning teams from the $n/2$ games. So, there are $n/2$ winners of Day 1, $n/2$ winners of Day 2, $\ldots$, and $n/2$ winners of Day $(n-1)$. Prove that no matter how the $\binom{n}{2}$ individual games turned out, it is always possible (after all of the games) to select one team who was a winner of Day 1, one team who was a winner of Day 2, $\ldots$, and one team who was a winner of Day $(n-1)$, such that we don't pick the same team twice. Note that since there are n teams in total, this selection will always leave exactly one team out.

4. Prove that every planar graph has a vertex of degree less than 6.