

## 21-228 Discrete Mathematics

### Assignment 5

Due Fri Mar 22, at start of class

**Notes:** Collaboration is permitted except in the writing stage. Also, please justify every numerical answer with an explanation.

1. Solve the recurrence  $a_n = 3a_{n-1} + 2$ , with initial condition  $a_0 = 0$ .
2. Consider the function  $f(z) = \frac{1}{\sqrt{1-4z}}$ . Find a nice formula for the coefficient of  $z^n$  when this is expanded as a power series about 0. That is, when it is expanded as  $f(z) = c_0z^0 + c_1z^1 + \dots$ , what is a general formula for  $c_n$ ? You may express your answer in terms of (integer) factorials and binomial coefficients of the form  $\binom{a}{b}$ , where  $a$  and  $b$  may depend on  $n$ , but are always non-negative integers (no matter what  $n$  is).
3. We learned in class that the  $n$ -th Catalan number  $C_n$  was the number of strings of length  $2n$  consisting of the characters '(' and ')', such that they were valid expressions. For example,  $C_3 = 5$ , as the five ways are  $()()()$ ,  $()(())$ ,  $((())()$ ,  $((()())$ , and  $((()))$ . Let  $D_n$  be the number of strings of length  $2n$  consisting of the characters '(', ')', '[', and ']', such that they are valid expressions. Now,  $(([]))$  is not a valid expression, because the underlined '(' is closed by the underlined ']', and of course, something like  $()\underline{)]}$  is still not valid, because the underlined ')' is closing nothing. On the other hand,  $(\underline{[]})$  is a valid expression.

Find and prove a general formula for  $D_n$ .

4. After the end of a round-robin math tournament among  $n$  students (in which every pair of students was matched head-to-head exactly once), it was observed that every student had won exactly the same number of games. Characterize all  $n$  for which this could have happened.

This means that you should describe a set  $S \subset \mathbb{Z}^+$ , and then:

- (a) for every  $n \in S$ , show that there is a way to choose the  $\binom{n}{2}$  outcomes of the head-to-head matches such that every student wins the same number of times; and also
- (b) for every  $n \notin S$ , prove that no matter how the  $\binom{n}{2}$  matches played out, it is impossible for every student to have the same number of wins.

For example, it is relatively easy to see that  $3 \in S$  because it's possible for Alice to beat Bob, Bob to beat Charlie, and Charlie to beat Alice, resulting in 1 win for each student. Also, it is easy to see that  $2 \notin S$  because if  $n = 2$ , then there is only one game, and it can only give the win to one of the students.

5. Suppose that an arrow is drawn on each edge of a cube, giving each edge a direction, in such a way that every vertex of the cube has at least one arrow coming out of it and at least one arrow going into it. (A cube has 6 faces, 8 vertices, and 12 edges, so there will be 12 arrows.) Prove that under these conditions, it is always possible to find a face of the cube such that the directions of the four boundary edges of that face go in a cycle.