21-228 Discrete Mathematics

Assignment 4

Due Fri Mar 1, at start of class

Notes: Collaboration is permitted except in the writing stage. Also, please justify every numerical answer with an explanation.

1. Consider the first 3^n rows of Pascal's triangle. These are the rows from $\binom{0}{0}$, which has single number in a row, to $\binom{3^n-1}{0}, \ldots, \binom{3^n-1}{3^n-1}$, which has 3^n numbers in a row. For each row, count the number of entries which are **not** divisible by 3. If the number of such entries in a row is a prime power (expressible as p^k for some prime p and some positive integer $k \geq 1$), then we say that the row is *cool*. How many of the first 3^n rows are cool? Find a general formula in terms of n. It's a nice formula, expressible using only arithmetic and power operations, and without any ellipses or summation ($\sum$) or product ($\prod$) notation.

   For example, if $n = 1$, then we are looking at the first three rows. The first row is just "1", and so it has 1 entry which is not divisible by 3. As 1 is not a prime power, this row is not cool. The second row is "1 1", with 2 entries that are not divisible by 3. Since 2 is a prime power, this row is cool. The third row is "1 2 1", with 3 entries (a prime power) non-divisible by 3, hence it is also cool. Therefore, if $n = 1$, then 2 out of the first 3^1 rows are cool. As a check for your formula, it is known that if $n = 4$, then 30 out of the first 3^4 rows are cool.

2. The Jacobsthal numbers are defined by the recursion $a_n = a_{n-1} + 2a_{n-2}$ with initial conditions $a_1 = 1$, $a_2 = 3$. Prove that

   $$a_n = \text{round}\left\{ \frac{2^{n+1}}{3} \right\},$$

   for every nonnegative integer n. Here, round(x) denotes the nearest integer to x, rounding up if x is a half-integer. For example, round$(1.1) = 1 =$ round(0.9) and round$(1.5) = 2$.

3. Find an explicit formula for the recursion defined by $a_n = 2a_{n-1} - 2a_{n-2}$ with initial conditions $a_0 = 0$ and $a_1 = 1$.

4. In class, we handled the case when matrices were diagonalizable. This exercise guides you through the case when the matrix is not!

   Find an explicit formula for the solution to the recurrence $a_n = 4a_{n-1} - 4a_{n-2}$, with initial conditions $a_0 = 0$ and $a_1 = 1$. Please use the following (rather than guessing the formula and using induction):

   - (Jordan Canonical Form.) There exists a 2×2 matrix P for which

     $$\begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix} = P \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} P^{-1}.$$

   - (Power of elementary Jordan block.) For any λ and any positive integer n:

     $$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}^n = \begin{bmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{bmatrix}.$$

5. Consider the generating function

$$\frac{1}{1-2x-x^2} = \sum_{n=0}^{\infty} a_n x^n.$$

Prove that for each integer $n \geq 0$,

$$a_n^2 + a_{n+1}^2 = a_{2n+2}.$$

Hint: Find a 2×2 matrix A such that

$$A^{n+2} = \begin{bmatrix} a_n & a_{n+1} \\ a_{n+1} & a_{n+2} \end{bmatrix},$$

and consider the top left entry of the matrix product $A^{n+2}A^{n+2}$.