21-228 Discrete Mathematics

Assignment 7

Due Mon Apr 24, at start of class

Notes: Collaboration is permitted except in the writing stage. Also, please justify every numerical answer with an explanation.

1. Is the following statement true for every positive integer n?

   Every graph consisting of two edge-disjoint Hamiltonian paths contains a Hamiltonian cycle.

2. Construct a random graph on n vertices as follows. Start with n vertices, with no edges. Then, for each of the $\binom{n}{2}$ pairs of vertices, flip a (very unfair) coin which is heads with probability $\frac{1}{n^{1.01}}$. If it is heads, put the edge between that pair of vertices; otherwise, do not.

   Let q_n be the probability that the resulting graph contains a triangle (three vertices a, b, c with edges ab, bc, and ca all present). Show that

$$\lim_{n\to\infty} q_n = 0\,.$$

3. Construct a random graph on n vertices as follows. Start with n vertices, with no edges. Then, for each of the $\binom{n}{2}$ pairs of vertices, flip a (very unfair) coin which is heads with probability $\frac{1}{n}$. If it is heads, put the edge between that pair of vertices; otherwise, do not. Prove that the probability of the graph being connected tends to zero as $n \to \infty$. (Much more is true.)

4. Let $R(t, t, t)$ be the smallest integer n such that every 3-coloring of the edges of K_n contains a monochromatic K_t. Prove that

   (a) $R(t, t, t) \le 27^t$, and

   (b) derive an exponential lower bound for the 3-color Ramsey number $R(t, t, t)$. That is, for each t, come up with an n_t, and a 3-coloring of the edges of K_{n_t} such that there is no monochromatic K_t, where n_t grows exponentially with t.

   Note: I can do this with $n_t = 3^{t/2}$ for every even t (which is sufficient), but it is OK if you prove this with a different n_t, as long as yours grows exponentially with t.

5. There are 100 cities in a country, called “1”, “2”, . . . , “100”. The government would like to connect them all with a network, and it costs $\max\{|i - j|, 4\}$ to lay a wire between cities “i” and “j”. (There is a minimum cost of 4 per wire.) Another company comes along, offering to create microwave links between city pairs at a cost of 2 each, with the catch that their technology can only connect city pairs of the form (“i”, “$2i$”). For example, it would cost 30 to lay a wire between cities 31 and 61, 4 to lay a wire between cities 31 and 32, and 2 to microwave-link 30 and 60. What is the minimum cost to create this connected network?