Extremal Combinatorics: Homework 3

Due Friday April 29, in class

1. Let $f(k, n)$ be the maximum number of edges in an n-vertex graph $G = (V, E)$ whose edge set is the union of k induced matchings. This means $E = E_1 \cup \ldots \cup E_k$, where each E_i is precisely $G[V_i]$ for some subset $V_i \subset V$, with bisections $V_i = V_i' \cup V_i''$ such that there are exactly $|V_i'|$ edges in each $G[V_i]$, and these edges all run between distinct pairs of vertices, one in V_i' and one in V_i''. Note in particular that if $E_i = G[V_i]$ is in the above partition, then G has no other edges with both endpoints in V_i.

   Prove that $f(n, n) = o(n^2)$, i.e., for every $\epsilon > 0$, there exists an N such that for all $n > N$, $f(n, n) \leq \epsilon n^2$.

2. Show that for every $\epsilon > 0$, there exist $c, N > 0$ such that the following holds for every $n > N$. Every n-vertex graph with at least $\left(\frac{1}{8} + \epsilon\right)n^2$ edges and independence number $\alpha < cn$ must contain a copy of K_4.

3. Let G be an n-vertex graph with the property that for every pair of distinct vertices $\{x, y\}$, the number of vertices z which are adjacent to both x and y is odd. Prove that n, the number of vertices of G, must also be odd.