14. Geometry

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1 Classical results

Triangle area. Let ABC be a triangle with side lengths $a = BC$, $b = CA$, and $c = AB$, and let r be its inradius and R be its circumradius. Let $s = \frac{a+b+c}{2}$ be its semiperimeter. Then its area is

$$sr = \sqrt{s(s-a)(s-b)(s-c)} = \frac{abc}{4R} = \frac{1}{2}ab\sin C\,.$$

Pick. The area of any polygon with integer vertex coordinates is exactly $I + \frac{B}{2} - 1$, where I is the number of lattice points in its interior, and B is the number of lattice points on its boundary.

Descartes. Let a, b, c, d be the radii of four mutually tangent circles. Let w, x, y, z be their inverses. Then

$$w^2 + x^2 + y^2 + z^2 = \frac{1}{2}(w + x + y + z)^2\,.$$

Archimedes. Take an orange, and pass it through a bread slicer, which has equally spaced blades. You get lots of differently shaped bits of orange peel (and some orange mess). The surface areas of the outsides of the peels are all the same! Wow!

2 Problems

1. Two circles with radii 1 and 2 are placed so that they are tangent to each other and a straight line. A third circle is nestled between them so that it is tangent to the first two circles and the line. Find the radius of the third circle.

2. Take a 5×5 square, and put 3-4-5 right triangles on its top and bottom sides, oriented such that they stick out of the square, and are 180-degree rotations of each other. Determine the distance between the right-angled corners of the 3-4-5 triangles.

3. The radius of the base of a right circular cone is 1. The vertex of the cone is V, and P is a point on the circumference of the base. The length of PV is 6 and the midpoint of PV is M. A piece of string is attached to M and wound tightly twice round the cone finishing at P. What is the length of the string?

4. The 3-dimensional vectors A, B, and C satisfy:

   $$A \times B = B \times C = C \times A \neq 0.$$

   Prove that $A + B + C = 0$.

5. Suppose that a disk of radius r is covered by m rectangular strips of width 2 each. Prove that $m \geq r$.

6. A ring of height h is obtained by digging a cylindrical hole through the center of a sphere. Prove that the volume of the ring depends only on h and not on the radius of the sphere.
7. Two convex polygons are placed one inside the other. Prove that the perimeter of the polygon that lies inside is smaller.
8. Find all finite sets S of at least three points in the plane such that for all distinct points $A, B \in S$, the perpendicular bisector of AB is an axis of symmetry for S.
9. A sequence of polygons is derived as follows. The first polygon is a regular hexagon of area 1. Thereafter each polygon is derived from its predecessor by joining two adjacent edge midpoints and cutting off the corner. Show that all the polygons have area greater than 1/3.
10. A cubical box with sides of length 7 has vertices at (0, 0, 0), (7, 0, 0), (0, 7, 0), (7, 7, 0), (0, 0, 7), (7, 0, 7), (0, 7, 7), (7, 7, 7). The inside of the box is lined with mirrors and from the point (0, 1, 2), a beam of light is directed to the point (1, 3, 4). The light then reflects repeatedly off the mirrors on the inside of the box. Determine how far the beam of light travels before it first returns to its starting point at (0, 1, 2).
11. Three distinct points with integer coordinates lie in the plane on a circle of radius $r > 0$. Show that two of these points are separated by a distance of at least $r^{1/3}$.

3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class. You are encouraged to collaborate with each other. Even if you do not solve a problem, please spend two hours thinking, and submit a list of your ideas.