12. Analysis

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1 Classical results

Harmonic Series. The sequence defined by

$$a_n = \left(\sum_{k=1}^{n} \frac{1}{k} \right) - \log n$$

converges to a real number.

2 Problems

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a surjective continuous function that takes any value at most twice. Prove that f is strictly monotone. Does this still follow if we only know that it takes any value at most three times?

2. Let $a_1, a_2, \ldots$ be a sequence such that

$$\lim_{n \to \infty} a_n - a_{n-2} = 0.$$

   Show that (a) it is not necessarily true that $\lim_{n \to \infty} a_n - a_{n-1} = 0$, but (b) it is always true that

$$\lim_{n \to \infty} \frac{a_n - a_{n-1}}{n} = 0.$$

3. Compute the antiderivative

$$\int (1 + 2x^2) e^{x^2} \, dx.$$

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $|f(x) - f(y)| \geq |x - y|$ for all $x, y \in \mathbb{R}$. Prove that f is surjective.