10. Combinatorics

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1 Classical results

Designs. There are $2n$ students at a school, for some integer $n \geq 2$. Each week n students go on a trip. After several trips the following condition was fulfilled: every two students were together on at least one trip. What is the minimum number of trips needed for this to happen?

Catalan numbers. Find a closed-form expression for the number of valid sequences containing n pairs of parantheses. For example, when $n = 2$, there are 2 valid sequences: ()() and (()). The sequence ())(is not valid.

Partitions. For every positive integer n, let $p(n)$ denote the number of ways to express n as a sum of positive integers. For instance, $p(4) = 5$ because

$$4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1.$$

Also, let $p(0) = 1$.

Prove that $p(n) - p(n-1)$ is the number of ways to express n as a sum of integers each of which is strictly greater than 1.

2 Problems

1. Consider a circular necklace with 2013 beads, each of which is painted either white or green. Call a painting "good" if, among any 21 successive beads, there is at least one green bead. Prove that the number of good paintings of a necklace is odd. **Note:** here, two paintings that differ on some beads, but can be obtained from each other by rotating or flipping the necklace, are counted as different paintings.

2. An alien race has three genders: male, female, and emale. A *married triple* consists of three persons, one from each gender, who all like each other. Any person is allowed to below to at most one married triple. A special feature of this race is that feelings are always mutual: if x likes y, then y likes x.

   The race is sending an expedition to colonize a planet. The expedition has n males, n females, and n emales. It is known that every expedition member likes at least k persons of each of the two other genders. The problem is to create as many married triples as possible to produce healthy offspring so the colony could grow and prosper.

   (a) Show that if n is even and $k = n/2$, then it might be impossible to create even one married triple.

   (b) Show that if $k \geq 3n/4$, then it is always possible to create n disjoint married triples, thus marrying all of the expedition members.

3. Given an integer $n > 1$, let S_n be the group of permutations of the numbers $1, 2, \ldots, n$. Two players, A and B, play the following game. Taking turns, they select elements (one element at a time) from the group S_n. It is forbidden to select an element that has already been selected. The game ends when the selected elements generate the whole group S_n. The player who made the last move loses the game. The first move is made by A. Which player has a winning strategy?

4. Let M be a set of $n \geq 4$ points in the plane, no three of which are collinear. Initially these points are connected with n segments so that each point in M is the endpoint of exactly two segments. Then, at each step, one may choose two segments AB and CD sharing a common interior point and replace them by the segments AC and BD if none of them is present at this moment. Prove that it is impossible to perform $n^3/4$ or more such moves.

5. The *distance* between any pair of vertices in a graph is the number of edges in the shortest path between them. The *diameter* of a graph is the maximum distance between any pair of vertices. A graph is called *diameter-2-critical* if it has diameter 2, but for every edge in the graph, the deletion of that edge would strictly increase the graph's diameter.

   Show that there is a diameter-2-critical graph with $n = 2015$ vertices, such that the sum of the squares of the degrees of the vertices is strictly greater than nm, where m is its number of edges.

3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class. You are encouraged to collaborate with each other. Even if you do not solve a problem, please spend two hours thinking, and submit a list of your ideas.