9. Linear Algebra

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1 Classical results

Vandermonde. The determinant of the matrix

$$\begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{pmatrix}$$

is

$$\prod_{1 \le i < j \le n} (x_j - x_i).$$

Parallelogram. Let v and w be vectors in $\mathbb{R}^n$, and let $\|v\|$ denote the length of v. Prove that:

$$\|v + w\|^2 + \|v - w\|^2 = 2\|v\|^2 + 2\|w\|^2.$$

Isometry via Polarization. Let the real $n \times n$ matrix A be an *isometry*, i.e., so that for all vectors $x \in \mathbb{R}^n$:

$$\|Ax\| = \|x\|.$$

Prove that this is equivalent to the statement that

$$\langle Ax, Ay \rangle = \langle x, y \rangle$$

for all $x, y \in \mathbb{R}^n$.

2 Problems

1. Let P be a projection, so that $P^2 = P$. If $c \neq 1$, compute $(I - cP)^{-1}$.
2. Calculate the value of the determinant of the 3×3 complex matrix X, provided that $\text{tr}(X) = 1$, $\text{tr}(X^2) = -3$, and $\text{tr}(X^3) = 4$. Here, $\text{tr}(A)$ denotes the *trace*, that is, the sum of the diagonal entries of the matrix A.
3. Prove that for any integers $x_1, x_2, \ldots, x_n$ and positive integers $k_1, k_2, \ldots, k_n$, the determinant of the matrix

$$\begin{pmatrix} x_1^{k_1} & x_1^{k_2} & \cdots & x_1^{k_n} \\ x_2^{k_1} & x_2^{k_2} & \cdots & x_2^{k_n} \\ \vdots & \vdots & \vdots & \vdots \\ x_n^{k_1} & x_n^{k_2} & \cdots & x_n^{k_n} \end{pmatrix}$$

is divisible by $(n!)$.

4. Let $\mathbf{A}$ and $\mathbf{B}$ be different $n \times n$ matrices with real entries. If $\mathbf{A}^3 = \mathbf{B}^3$ and $\mathbf{A}^2\mathbf{B} = \mathbf{B}^2\mathbf{A}$, can $\mathbf{A}^2 + \mathbf{B}^2$ be invertible?

5. Let n be a positive integer and let $x_1, \ldots, x_n$ be n nonzero points in the plane. Suppose $\langle x_i, x_j \rangle$ (scalar or dot product) is a rational number for all i, j. Let S denote all points of the plane of the form $\sum_{i=1}^{n} a_i x_i$ where the a_i are integers. A closed disk of radius R and center P is the set of points at distance at most R from P (includes the points distance R from P). Prove that there exists a positive number R and closed disks $D_1, D_2, \ldots$ of radius R such that (a) Each disk contains exactly two points of S; (b) Every point of S lies in at least one disk; (c) Two distinct disks intersect in at most one point.

6. Let A and B be 2×2 matrices with integer entries, such that $AB = BA$ and $\det B = 1$. Prove that if $\det(A^3 + B^3) = 1$, then A^2 is the zero matrix.

7. Let A be an $n \times n$ matrix. Prove that there exists an $n \times n$ matrix B such that $ABA = A$.

8. For an $n \times n$ matrix A, denote by $\phi_k(A)$ the k-th symmetric polynomial in the eigenvalues λ_1, λ_2, $\ldots$, λ_n of A,

$$\phi_k(A) = \sum_{i_1 < i_2 < \cdots < i_k} \lambda_{i_1} \lambda_{i_2} \ldots \lambda_{i_k}.$$

   For example, $\phi_1(A)$ is the trace and $\phi_n(A)$ is the determinant. Prove that for two $n \times n$ matrices A and B, $\phi_k(AB) = \phi_k(BA)$ for all $k = 1, 2, \ldots, n$.

3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class. You are encouraged to collaborate with each other. Even if you do not solve a problem, please spend two hours thinking, and submit a list of your ideas.