8. Recursions

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1 Classical results

Tilings. Determine the number of ways to tile a 1×10 strip using only 1×1 or 1×2 tiles.

Catalan numbers. Find a closed-form expression for the number of valid sequences containing n pairs of parantheses. For example, when $n = 2$, there are 2 valid sequences: ()() and (()). The sequence ())(is not valid.

Fibonacci formula. For all positive integers n, the n-th Fibonacci number is the closest integer to $\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)$.

2 Problems

1. Go to **Scaife 125** at 8:45am on Saturday.
2. Prove that for any $n \geq 1$, a $2^n \times 2^n$ checkerboard with any 1×1 square removed can be tiled by L-shaped triominoes.
3. How many sequences of 1's and 3's sum to 16? (Examples of such sequences are $\{1, 3, 3, 3, 3, 3\}$ and $\{1, 3, 1, 3, 1, 3, 1, 3\}$.)
4. A sequence is defined by $a_0 = -1$, $a_1 = 0$, and
$$a_{n+1} = a_n^2 - (n+1)^2 a_{n-1} - 1$$
for all positive integers n. Find a_{100}.
5. A type 1 sequence is a sequence with each term 0 or 1 which does not have 0, 1, 0 as consecutive terms. A type 2 sequence is a sequence with each term 0 or 1 which does not have 0, 0, 1, 1 or 1, 1, 0, 0 as consecutive terms. Show that there are twice as many type 2 sequences of length $n+1$ as type 1 sequences of length n.
6. For each positive integer n, let S_n denote the total number of squares in an $n \times n$ square grid. Thus $S_1 = 1$ and $S_2 = 5$, because a 2×2 square grid has four 1×1 squares and one 2×2 square. Find a recurrence relation for S_n, and use it to calculate the total number of squares on a chess board (i.e. determine S_8).
7. How about the number of rectangles?
8. Let F_n be the Fibonacci sequence with $F_0 = F_1 = 1$. Evaluate
$$\sum_{n=1}^{\infty} \frac{1}{F_{n-1} F_{n+1}}.$$

9. For n a positive integer, define $f_1(n) = n$, and then for each i, let $f_{i+1}(n) = f_i(n)^{f_i(n)}$. Determine $f_{100}(75) \bmod 17$.

10. Define the function $f : (0,1) \to (0,1)$ by

$$f(x) = \begin{cases} x + \frac{1}{2} & \text{if } x < \frac{1}{2}, \\ x^2 & \text{if } x \geq \frac{1}{2}. \end{cases}$$

Let a and b be two real numbers such that $0 < a < b < 1$. We define the sequences a_n and b_n by $a_0 = a$, $b_0 = b$, and $a_n = f(a_{n-1})$, $b_n = f(b_{n-1})$ for $n > 0$. Show that there exists a positive integer n such that

$$(a_n - a_{n-1})(b_n - b_{n-1}) < 0.$$

3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class. You are encouraged to collaborate with each other. Even if you do not solve a problem, please spend two hours thinking, and submit a list of your ideas.