7. Convergence

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CMU Putnam Seminar, Fall 2015

1 Classical results

Harmonic series. Without using Calculus, show that $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Refinement. Is the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)}$$

convergent or divergent?

Alternating series. Let (a_n) be a monotonic decreasing sequence of positive real numbers. Then the series $\sum_{n=1}^{\infty} (-1)^n a_n$ is convergent.

2 Problems

1. Let (a_n) be a monotonic decreasing sequence of positive real numbers with limit 0 (so $a_1 \geq a_2 \geq \cdots \geq 0$). Let (b_n) be a rearrangement of the sequence such that for every non-negative integer m, the terms b_{3m+1}, b_{3m+2}, b_{3m+3} are a rearrangement of the terms a_{3m+1}, a_{3m+2}, a_{3m+3} (thus, for example, the first 6 terms of the sequence (b_n) could be $a_3, a_2, a_1, a_4, a_6, a_5$). Prove or give a counterexample to the following statement: the series $\sum_{n=1}^{\infty} (-1)^n b_n$ is convergent.

2. Determine whether the series

$$\sum_{n=2}^{\infty} \frac{1}{n^{1+(\log\log n)^{-2}}}$$

is convergent or divergent. (The logarithm is in base e.)

3. Let (a_n) be a sequence of positive real numbers such that $\lim_{n\to\infty} a_n = 0$. Prove that $\sum_{n=1}^{\infty} \left| 1 - \frac{a_{n+1}}{a_n} \right|$ is divergent.

4. Find

$$\lim_{x\to\infty} (2x)^{1+\frac{1}{2x}} - x^{1+\frac{1}{x}} - x.$$

5. Let $(a_n)_{n>1}$ be an infinite sequence with $a_n > 0$ for all n. For $n > 1$, let b_n denote the geometric mean of $a_1, \ldots, a_n$, that is, $\sqrt[n]{a_1 \cdots a_n}$. Suppose $\sum_{n=1}^{\infty} a_n$ is convergent. Prove that $\sum_{n=1}^{\infty} b_n^2$ is also convergent.

6. Assume that $(a_n)_{n\geq 1}$ is an increasing sequence of positive real numbers such that $\lim \frac{a_n}{n} = 0$. Must there exist infinitely many positive integers n such that $a_{n-i} + a_{n+i} < 2a_n$ for $i = 1, 2, \ldots, n-1$?

7. Define a sequence by $a_1 = 1$, $a_2 = \frac{1}{2}$, and $a_{n+2} = a_{n+1} - \frac{a_n a_{n+1}}{2}$ for every positive integer n. Find

$$\lim_{n\to\infty} na_n.$$

8. Let $\sum_{n=1}^{\infty} a_n$ be a convergent series of positive terms (so $a_i > 0$ for all i), and define $b_n = \frac{1}{n a_n^2}$. Prove that

$$\sum_{n=1}^{\infty} \frac{n}{b_1 + b_2 + \cdots + b_n}$$

is convergent.

3   Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class. You are encouraged to collaborate with each other. Even if you do not solve a problem, please spend two hours thinking, and submit a list of your ideas.