6. Inequalities

Po-Shen Loh

CMU Putnam Seminar, Fall 2015

1 Classical results

AM-GM. For any non-negative reals $x_1, \ldots, x_n$,

$$\sqrt[n]{x_1 x_2 \cdots x_n} \le \frac{x_1 + \cdots + x_n}{n} .$$

Rearrangement. For any reals $x_1 \le x_2 \le \cdots \le x_n$ and $y_1 \le y_2 \le \cdots \le y_n$, and any reordering $y_{\sigma(1)}, y_{\sigma(2)}, \ldots, y_{\sigma(n)}$,

$$x_1 y_n + x_2 y_{n-1} + \cdots + x_n y_1 \le x_1 y_{\sigma(1)} + x_2 y_{\sigma(2)} + \cdots x_n y_{\sigma(n)} \le x_1 y_1 + x_2 y_2 + \cdots + x_n y_n .$$

Cauchy-Schwarz. For any reals $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$,

$$(x_1 y_1 + x_2 y_2 + \cdots + x_n y_n)^2 \le (x_1^2 + \cdots + x_n^2)(y_1^2 + \cdots + y_n^2) .$$

Jensen. For any convex function f, and any reals $x_1, \ldots, x_n$,

$$f\left(\frac{x_1 + \cdots + x_n}{n}\right) \le \frac{f(x_1) + \cdots + f(x_n)}{n} .$$

2 Problems

1. Several gas stations are located along a circular road. Among them, there is just enough gas for one car to complete a single trip around the circle. Is it always true that there is always a place where you can start, so that your car can make it all the way around once?

2. A set of $n > 3$ real numbers has sum at least n and the sum of the squares of the numbers is at least n^2. Show that the largest positive number is at least 2.

3. We have 2^m sheets of paper, with the number 1 written on each of them. We perform the following operation. In every step we choose two distinct sheets; if the numbers on the two sheets are a and b, then we erase these numbers and write the number $a + b$ on both sheets. Prove that after $m2^{m-1}$ steps, the sum of the numbers on all the sheets is at least 4^m.

4. Let m and n be positive integers. Let $a_1, a_2, \ldots, a_m$ be distinct elements of $\{1, 2, \ldots, n\}$ such that whenever $a_i + a_j \le n$ for some i, j (possibly the same) we have $a_i + a_j = a_k$ for some k. Prove that:

$$\frac{a_1 + \cdots + a_m}{m} \ge \frac{n+1}{2} .$$

5. Let $a_1, a_2, \ldots, a_n$ be a sequence of real numbers, and let m be a fixed positive integer less than n. We say an index k with $1 \le k \le n$ is *good* if there exists some ℓ with $1 \le \ell \le m$ such that

$$a_k + a_{k+1} + \cdots + a_{k+\ell-1} \ge 0,$$

where the indices are taken modulo n. Let T be the set of all good indices. Prove that

$$\sum_{k \in T} a_k \ge 0.$$

6. Can an arc of a parabola inside a circle of radius 1 have a length greater than 4?

7. For a sequence $x_1, x_2, \ldots, x_n$ of real numbers, we define its *price* as

$$\max_{1 \le i \le n} |x_1 + \cdots + x_i|.$$

Given n real numbers, Dave and George want to arrange them into a sequence with a low price. Diligent Dave checks all possible ways and finds the minimum possible price D. Greedy George, on the other hand, chooses x_1 such that $|x_1|$ is as small as possible; among the remaining numbers, he chooses x_2 such that $|x_1 + x_2|$ is as small as possible, and so on. Thus, in the i-th step he chooses x_i among the remaining numbers so as to minimize the value of $|x_1 + x_2 + \cdots + x_i|$. In each step, if several numbers provide the same value, George chooses one at random. Finally he gets a sequence with price G.

Find the least possible constant c such that for every positive integer n, for every collection of n real numbers, and for every possible sequence that George might obtain, the resulting values satisfy the inequality $G \le cD$.

8. (From DNA nanotechnology.) Define the functions

$$\chi(K) = \frac{(a+1)cK + 1}{2cK} - \frac{\sqrt{(a-1)^2c^2K^2 + 2(a+1)cK + 1}}{2cK}$$
$$Q(K) = \frac{\chi(zK)}{\chi(K)}.$$

Prove that for any $a, z \ge 1$ and $c, K > 0$, we always have $Q(K) \le z$.

3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class. You are encouraged to collaborate with each other. Even if you do not solve a problem, please spend two hours thinking, and submit a list of your ideas.