1. Introduction

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1 Classical results

1. Recall that
$$e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots \approx 2.718\ldots$$
Prove that e is irrational.

2. Prove that for every positive integer n, there exists a finite set of points in the plane such that for every point of the set there exist exactly n other points of the set at distance equal to 1 from that point.

3. A *Hadamard Matrix* is an $n \times n$ square matrix, all of whose entries are $+1$ or -1, such that every pair of distinct rows is orthogonal. In other words, if the rows are considered to be vectors of length n, then the dot product between any two distinct row-vectors is zero. Show that infinitely many Hadamard Matrices exist.

4. Hadamard Conjecture (open): for every positive integer k, there is a Hadamard Matrix of order $4k$. The first unknown case is $4k = 668$.

5. (Hamming codes.) A license plate has seven binary digits (0 or 1), and may have leading zeros. If two plates must always differ in at least three places, what is the largest number of plates that is possible?

2 Problems

1. Show that any convex polyhedron has two faces with the same number of edges.

2. Let $n > 1$ be an arbitrary positive integer, and let k be the number of positive prime numbers less than or equal to n. Select $k+1$ positive integers such that none of them divides the product of all the others. Prove that there exists a number among the chosen $k+1$ that is bigger than n.

3. Every point in $\mathbb{R}^3$ is colored either red, green, or blue. Prove that one of the colors attains all distances, i.e., every positive real number represents the distance between two points of this color.

4. Prove that for any positive integer $n \geq 2$ there is a positive integer m that can be written simultaneously as a sum of 2, 3, $\ldots$, n squares of nonzero integers.

5. Prove that for every positive integer n, there exists an n-digit number divisible by 5^n, all of whose digits are odd.

6. An $n \times n$ matrix whose entries come from the set $S = \{1, 2, \ldots, 2n-1\}$ is called a *silver matrix* if, for each $i = 1, 2, \ldots, n$, the i-th row and the i-th column together contain all elements of S. Show that:

   (a) there is no silver matrix for $n = 1997$;

   (b) silver matrices exist for infinitely many values of n.

7. There are $n > 1$ lamps L_0, L_1, ..., L_{n-1} in a circle. We use L_{n+k} to mean L_k. A lamp is at all times either on or off. Initially they are all on. Perform steps s_0, s_1, ... as follows: at step s_i, if L_{i-1} is lit, then switch L_i from on to off or vice versa, otherwise do nothing. Show that there is a positive integer $M(n)$ such that after $M(n)$ steps all the lamps are on again;

8. Find the smallest positive integer n such that if n squares of a 1000×1000 chessboard are colored, then there will exist three colored squares whose centers form a right triangle with sides parallel to the edges of the board.

9. A license plate has six digits from 0 to 9 and may have leading zeros. If two plates must always differ in at least two places, what is the largest number of plates that is possible?

10. Prove that for any $n \geq 1$, a $2^n \times 2^n$ checkerboard with any 1×1 square removed can be tiled by L-shaped triominoes.

11. Let n be an odd integer greater than 1, and let $k_1, k_2, \ldots, k_n$ be given integers. For each of the $n!$ permutations $a = (a_1, a_2, \ldots, a_n)$ of $1, 2, \ldots, n$, let

$$S(a) = \sum_{i=1}^{n} k_i a_i \,.$$

Prove that there are two different permutations b and c such that $n!$ is a divisor of $S(b) - S(c)$.

12. Call a real-valued function *very convex* if:

$$\frac{f(x) + f(y)}{2} \geq f\left(\frac{x+y}{2}\right) + |x - y|$$

holds for all real numbers x and y. Prove that no *very convex* function exists.

13. Let S be a finite set of at least two points in the plane. Assume that no three points of S are collinear. A *windmill* is a process that starts with a line ℓ going through a single point $P \in S$. The line rotates clockwise about the pivot P until the first time that the line meets some other point Q belonging to S. This point Q takes over as the new pivot, and the line now rotates clockwise about Q, until it next meets a point of S. This process continues indefinitely. Show that we can choose a point P in S and a line ℓ going through P such that the resulting windmill uses each point of S as a pivot infinitely many times.

3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class. You are encouraged to collaborate with each other. Even if you do not solve a problem, please spend two hours thinking, and submit a list of your ideas.