21-228 Discrete Mathematics

Assignment 5

Due Wed Mar 25, at start of class

Notes: Collaboration is permitted except in the writing stage. Also, please justify every numerical answer with an explanation.

1. Solve the recurrence $a_n = 3a_{n-1} + 2$, with initial condition $a_0 = 0$.

2. Consider the function $f(z) = \frac{1}{\sqrt{1-4z}}$. Find a nice formula for the coefficient of z^n when this is expanded as a power series about 0. That is, when it is expanded as $f(z) = c_0 z^0 + c_1 z^1 + \cdots$, what is a general formula for c_n? You may express your answer in terms of (integer) factorials and binomial coefficients of the form $\binom{a}{b}$, where a and b may depend on n, but are always non-negative integers (no matter what n is).

3. We learned in class that the n-th Catalan number C_n was the number of strings of length $2n$ consisting of the characters '(' and ')', such that they were valid expressions. For example, $C_3 = 5$, as the five ways are ()()(), ()(()), (())(), (()()), and ((())). Let D_n be the number of strings of length $2n$ consisting of the characters '(', ')', '[', and ']', such that they are valid expressions. Now, (([]]) is not a valid expression, because the underlined '(' is closed by the underlined ']', and of course, something like ())[][is still not valid, because the underlined ')' is closing nothing. On the other hand, ([]()) is a valid expression.

   Find and prove a general formula for D_n.

4. After the end of a round-robin math tournament among n students (in which every pair of students was matched head-to-head exactly once), it was observed that every student had won exactly the same number of games. Characterize all n for which this could have happened.

   This means that you should describe a set $S \subset \mathbb{Z}^+$, and then:

   (a) for every $n \in S$, show that there is a way to choose the $\binom{n}{2}$ outcomes of the head-to-head matches such that every student wins the same number of times; and also

   (b) for every $n \notin S$, prove that no matter how the $\binom{n}{2}$ matches played out, it is impossible for every student to have the same number of wins.

   For example, it is relatively easy to see that $3 \in S$ because it's possible for Alice to beat Bob, Bob to beat Charlie, and Charlie to beat Alice, resulting in 1 win for each student. Also, it is easy to see that $2 \notin S$ because if $n = 2$, then there is only one game, and it can only give the win to one of the students.

5. Suppose that an arrow is drawn on each edge of a cube, giving each edge a direction, in such a way that every vertex of the cube has at least one arrow coming out of it and at least one arrow going into it. (A cube has 6 faces, 8 vertices, and 12 edges, so there will be 12 arrows.) Prove that under these conditions, it is always possible to find a face of the cube such that the directions of the four boundary edges of that face go in a cycle.