15. Higher math

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1 Well-known results

Sign of a permutation. Given a permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$, its *number of inversions* $\iota(\sigma)$ is the number of pairs (i, j) such that $i < j$ but $\sigma(i) > \sigma(j)$. The sign of σ is $(-1)^{\iota(\sigma)}$.

2 Problems

1. A cube of side length 3 is divided into 27 unit cubes. The unit cubes are arbitrarily labeled 1 to 27 (each cube is given a different number). A move consists of swapping the cube labeled 27 with one of its (up to) 6 neighbors. Is it possible to find a finite sequence of moves at the end of which cube 27 is in its original position, but for each $n \in \{1, 2, \ldots, 26\}$, cube n has moved to the position originally occupied by $27 - n$?

2. Let G be a finite set of real $n \times n$ matrices $\{M_i\}$, $1 \le i \le r$, which form a group under matrix multiplication. Suppose that $\sum_{i=1}^{r} \mathrm{tr}(M_i) = 0$, where $\mathrm{tr}(A)$ denotes the trace of the matrix A. Prove that $\sum_{i=1}^{r} M_i$ is the $n \times n$ zero matrix.

3. Let $ABCDE$ be a convex pentagon with equal angles, all of whose sides have rational length. Prove that $ABCDE$ is a regular pentagon.

4. Prove that there exists a constant $c > 0$ such that in every nontrivial finite group G there exists a sequence of length at most $c \ln |G|$ with the property that each element of G equals the product of some subsequence. (The elements of G in the sequence are not required to be distinct. A *subsequence* of a sequence is obtained by selecting some of the terms, not necessarily consecutive, without reordering them; for example, $4, 4, 2$ is a subsequence of $2, 4, 6, 4, 2$, but $2, 2, 4$ is not.)

3 Homework

Please come to Scaife Hall, room 125, at 9:40am on Saturday, December 6.