8. Recursions

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1 Well-known statements

Fibonacci. The Fibonacci sequence is defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$. The $(n+1)$-st Fibonacci number F_{n+1} equals the number of ways to tile a $1 \times n$ board with 1×1 squares and 1×2 dominoes.

Cassini. The Fibonacci numbers satisfy

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n.$$

1/89. What's going on?

$$\frac{1}{89} = 0.01123595595\ldots$$

2 Problems

1. Let $x_0 = 1$, and for each $n \geq 0$, let $x_{n+1} = x_n + \frac{1}{x_n}$. Prove that $x_n \to \infty$.

2. The Fibonacci numbers satisfy $F_n^2 + F_{n+1}^2 = F_{2n+1}$.

3. How many sequences of 1's and 3's sum to 16? (Examples of such sequences are $\{1, 3, 3, 3, 3, 3\}$ and $\{1, 3, 1, 3, 1, 3, 1, 3\}$.)

4. A computer is programmed to randomly generate a string of six symbols using only the letters A, B, C. What is the probability that the string will not contain three consecutive A's?

5. Let $a_3 = \frac{2+3}{1+6}$, and for each $n \geq 4$, let

$$a_n = \frac{n + a_{n-1}}{1 + na_{n-1}}.$$

   Find a_{1995}.

6. Let n be a positive integer. A bit string of length n is a sequence of n numbers consisting of 0's and 1's. Let $f(n)$ denote the number of bit strings of length n in which every 0 is surrounded by 1's. (Thus for $n = 5$, 11101 is allowed, but 10011 and 10110 are not allowed, and we have $f(3) = 2$, $f(4) = 3$.) Prove that $f(n) < 1.7^n$ for all n.

7. Let x be a real number strictly between 0 and 1. For each positive integer n, define $f_n(t) = t + \frac{t^2}{n}$, and let

$$a_n = f_n(f_n(\ldots f_n(x))\ldots).$$

   Determine $\lim_{n\to\infty} a_n$.

3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class. You are encouraged to collaborate with each other. Even if you do not solve a problem, please spend two hours thinking, and submit a list of your ideas.