

## 4. Calculus

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### 1 Well-known statements

**Gaussian.**  $\int_{-\infty}^{\infty} e^{-x^2} = \sqrt{\pi}$ .

**Archimedes' Principle.** If you take a (perfectly spherical) orange, and slice it with a bagel slicer (with blades 2 cm apart), where both blades cut the orange, the surface area of peel you obtain is exactly the same no matter where along the orange you slice.

**Volume of torus.** The volume of a torus is  $(\pi r^2)(2\pi R)$ , where  $r$  is the radius of the circular cross section, and  $R$  is the distance from the center of the torus to the center of a circular cross section.

### 2 Problems

1. Determine  $f'(x)$ , if  $f(x) = \left[ \int_0^{x^2} e^{-x^2} \right]^2$ .

2. Let  $C$  be the unit circle  $x^2 + y^2 = 1$ . A point  $P$  is chosen randomly on the circumference  $C$  and another point  $Q$  is chosen randomly from the interior of  $C$  (these points are chosen independently and uniformly over their domains). Let  $R$  be the rectangle with sides parallel to the  $x$  and  $y$ -axes with diagonal  $PQ$ . What is the probability that no point of  $R$  lies outside of  $C$ ?

3. Find all real functions  $f$  for which  $\int_0^x f(t)dt = \frac{1}{2}xf(x)$ .

4. Suppose that  $f : [0, 1] \rightarrow \mathbb{R}$  has a continuous derivative and that  $\int_0^1 f(x)dx = 0$ . Prove that for every  $\alpha \in (0, 1)$ ,

$$\left| \int_0^\alpha f(x)dx \right| \leq \frac{1}{8} \max_{0 \leq x \leq 1} |f'(x)|.$$

5. Find the volume of the region of points  $(x, y, z)$  such that

$$(x^2 + y^2 + z^2 + 8)^2 \leq 36(x^2 + y^2).$$

6. Let  $P$  be a convex polygon, let  $Q$  be the interior of  $P$ , and let  $S = P \cup Q$ . Let  $p$  be the perimeter of  $P$  and let  $A$  be its area. Given any point  $(x, y)$ , let  $d(x, y)$  be the distance from  $(x, y)$  to the nearest point of  $S$ . Find constants  $\alpha$ ,  $\beta$ , and  $\gamma$  such that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-d(x,y)} dx dy = \alpha + \beta p + \gamma A.$$

7. Evaluate

$$\int_0^{\infty} t^{-1/2} e^{-1985(t+t^{-1})} dt.$$

8. Let  $V$  be the pyramidal region  $x, y, z \geq 0, x + y + z \leq 1$ . Evaluate

$$\int_V xy^9z^8(1-x-y-z)^4 dx dy dz.$$

9. Find all continuous functions  $f : [0, \infty) \rightarrow \mathbb{R}$  such that (i) for every  $x > 0$ ,  $f(x) > 0$ , and (ii) for all  $a > 0$ , the centroid of the region under the curve  $y = f(x)$  between  $0 \leq x \leq a$  has  $y$ -coordinate equal to the average value of  $f(x)$  on  $[0, a]$ .

### 3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class. You are encouraged to collaborate with each other. Even if you do not solve a problem, please spend two hours thinking, and submit a list of your ideas.