2. Polynomials

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1 Well-known statements

Limited roots. A polynomial of degree n has exactly n (complex) roots, counted with multiplicity.

Complete factorization. If the n roots of a degree-n polynomial $p(z)$ are $r_1, \ldots, r_n$, then we can express $p(z)$ as $a(z - r_1) \cdots (z - r_n)$.

Vieta's formulas. If $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$, then the product of the roots is $(-1)^n a_0$, and the sum of the roots is $-a_{n-1}$.

Uniqueness. For each nonnegative integer n, if two polynomials $p(x) = a_n x^n + \cdots + a_0$ and $q(x) = b_n x^n + \cdots + b_0$ are equal for $n+1$ distinct values of x, then all of their coefficients are equal, and they are the same polynomial.

Lagrange interpolation. If $p(x)$ is a polynomial of degree n, and we have real numbers $x_1, \ldots, x_{n+1}$ and $y_1, \ldots, y_{n+1}$ such that every $p(x_i) = y_i$, then there is an explicit formula for the polynomial $p(x)$.

2 Problems

1. If 3 distinct points on the curve $y = x^3$ are collinear, then the sum of the x-coordinates of those 3 points equals 0. There's actually a similar Putnam problem: show that if 4 distinct points on the curve $y = 2x^4 + 7x^3 + 3x - 5$ are collinear, then their average x-coordinate is some constant k; determine k.

2. Given any n real pairs $(x_1, y_1), \ldots, (x_n, y_n)$, with all x_i distinct, prove that there is a polynomial P such that $P(x_i) = y_i$ for all of those pairs, and also all roots of P are real.

3. Let $P(z)$ be a polynomial with complex coefficients. Prove that $P(z)$ is an even function if and only if there exists a polynomial $Q(z)$ with complex coefficients satisfying $P(z) = Q(z)Q(-z)$.

4. Describe all ordered pairs (a, b) of real numbers such that both (possibly complex) roots of $z^2 + az + b = 0$ satisfy $|z| < 1$.

5. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ has the property that for each fixed x, the function $g_x(y) = f(x, y)$ is a polynomial in y, and for each fixed y, the function $h_y(x) = f(x, y)$ is a polynomial in x. Prove that $f(x, y)$ must be a polynomial in x and y, i.e., that there is a finite n, and a finite collection of real numbers $a_{j,k}$ with $0 \le j, k \le n$, such that $f(x, y) = \sum_{j=0}^{n} \sum_{k=0}^{n} a_{j,k} x^j y^k$.

6. Prove that there is a function $f : \mathbb{Q}^2 \to \mathbb{Q}$ with the above property, but which is not a polynomial in x and y.

7. Invent a single (binary) operation $\star$ such that for every real numbers a and b, the operations $a + b$, $a - b$, $a \times b$, and $a \div b$ can be created by applying just $\star$ (possibly many times), starting with just a's and b's.

3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class. You are encouraged to collaborate with each other. Even if you do not solve a problem, please spend two hours thinking, and submit a list of your ideas.