Putnam Σ.15

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1 Problems

Putnam 2009/A1. Let f be a real-valued function on the plane such that for every square $ABCD$ in the plane, $f(A)+f(B)+f(C)+f(D)=0$. Does it follow that $f(P)=0$ for all points P in the plane?

Putnam 2009/A2. Functions f, g, h are differentiable on some open interval around 0 and satisfy the equations and initial conditions

$$\begin{aligned} f' &= 2f^2gh + \frac{1}{gh}, \quad f(0)=1, \\ g' &= fg^2h + \frac{4}{fh}, \quad g(0)=1, \\ h' &= 3fgh^2 + \frac{1}{fg}, \quad h(0)=1. \end{aligned}$$

Find an explicit formula for $f(x)$, valid in some open interval around 0.

Putnam 2009/A3. Let d_n be the determinant of the $n \times n$ matrix whose entries, from left to right and then from top to bottom, are $\cos 1, \cos 2, \ldots, \cos(n^2)$. For example, d_3 is the determinant of the matrix

$$\begin{pmatrix} \cos 1 & \cos 2 & \cos 3 \\ \cos 4 & \cos 5 & \cos 6 \\ \cos 7 & \cos 8 & \cos 9 \end{pmatrix}.$$

Note: The argument of cosine is always in radians, not degrees. Evaluate $\lim_{n\to\infty} d_n$.

Putnam 2009/A4. Let S be a set of rational numbers such that

(a) $0 \in S$;

(b) If $x \in S$, then $x+1 \in S$ and $x-1 \in S$; and

(c) If $x \in S$ and $x \neq \{0,1\}$, then $\frac{1}{x(x-1)} \in S$.

Must S contain all rational numbers?

Putnam 2009/A5. Is there a finite abelian group G such that the product of the orders of all its elements is 2^{2009}?

Putnam 2009/A6. Let $f : [0,1]^2 \to \mathbb{R}$ be a continuous function on the closed unit square such that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and are continuous on the interior $(0,1)^2$. Let $a = \int_0^1 f(0,y)dy$, $b = \int_0^1 f(1,y)dy$, $c = \int_0^1 f(x,0)dx$, and $d = \int_0^1 f(x,1)dx$. Prove or disprove: There must be a point (x_0, y_0) in $(0,1)^2$ such that

$$\frac{\partial f}{\partial x}(x_0,y_0) = b-a \quad \text{and} \quad \frac{\partial f}{\partial y}(x_0,y_0) = d-c.$$