Putnam Σ.11

Po-Shen Loh

3 November 2013

1 Problems

Putnam 1987/B4. Let $(x_1, y_1) = (0.8, 0.6)$ and let $x_{n+1} = x_n \cos y_n - y_n \sin y_n$ and $y_{n+1} = x_n \sin y_n + y_n \cos y_n$ for $n = 1, 2, 3, \ldots$. For each of $\lim_{n\to\infty} x_n$ and $\lim_{n\to\infty} y_n$, prove that the limit exists and find it or prove that the limit does not exist.

Putnam 1987/B5. Let O_n be the n-dimensional vector $(0, 0, \cdots, 0)$. Let M be a $2n \times n$ matrix of complex numbers such that whenever $(z_1, z_2, \ldots, z_{2n})M = O_n$, with complex z_i, not all zero, then at least one of the z_i is not real. Prove that for arbitrary real numbers $r_1, r_2, \ldots, r_{2n}$, there are complex numbers $w_1, w_2, \ldots, w_n$ such that

$$\text{re}\left[M\begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}\right] = \begin{pmatrix} r_1 \\ \vdots \\ r_{2n} \end{pmatrix}.$$

(Note: if C is a matrix of complex numbers, $\text{re}(C)$ is the matrix whose entries are the real parts of the entries of C.)

Putnam 1987/B6. Let F be the field of p^2 elements, where p is an odd prime. Suppose S is a set of $(p^2 - 1)/2$ distinct nonzero elements of F with the property that for each $a \neq 0$ in F, exactly one of a and $-a$ is in S. Let N be the number of elements in the intersection $S \cap \{2a : a \in S\}$. Prove that N is even.

IMC 1996/B1 (Bonus random analysis problem - work on if finished early). Let $f : [0, 1] \to [0, 1]$ be a continuous function. Let $x_1, x_2, \ldots$ be a sequence satisfying $x_{n+1} = f(x_n)$ for all $n \geq 1$, and suppose that

$$\lim_{n\to\infty} (x_{n+1} - x_n) = 0\,.$$

Prove that the entire sequence $x_1, x_2, \ldots$ converges.