15. Some algebra

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1 Famous results

Inverses are unique. If A, B, C are finite-dimensional square matrices such that $AB = I$ and $BC = I$, then $A = C$.

Commuting matrices. If A and B are finite-dimensional square matrices such that $AB = I$, then $BA = I$.

2 Problems

1. When and where is the Putnam?
2. If $\{(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)\}$ are orthonormal, then $\{(x_1, x_2, x_3), (y_1, y_2, y_3), (z_1, z_2, z_3)\}$ are also orthonormal.
3. Let S be a set with a binary operation "$*$", such that both

   (i) $a * a = a$ for all $a \in S$, and

   (ii) $(a * b) * c = (b * c) * a$ for all $a, b, c \in S$.

   Show that $*$ is associative and commutative.
4. Suppose that a standard, fair, 6-sided die is rolled 10 times. The sum of the numbers rolled will be somewhere between 10 and 60 inclusive. What is the probability that the sum will be divisible by 5?
5. Let G be a finite group. Show that it is impossible to find two proper subgroups A and B such that $A \cup B = G$. (A proper subgroup is a subgroup which is not the entire group.)
6. Let G be a finite group with n elements, and suppose that A is a set of more than $n/2$ elements of G. Prove that for every $g \ni G$, there are $x, y \in A$ (not necessarily distinct) such that $g = xy$.
7. Let $*$ be a binary operation on a set of n elements, which is commutative but not necessarily associative, and such that $a * b$ is always either a or b. Let the score of a be equal to the number of b's such that $a * b = a$. Then, there is a formula which guarantees that the score of the result is always at least $\lfloor \log_2 n \rfloor$. Actually, there's a formula that guarantees a resulting score of at least $\Theta(\sqrt{n})$.

3 No homework

Please do not submit write-ups for any problems. There is no homework for next week. There is no next week. Do not pass Go, do not collect $200.