13. Analysis

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1 Famous results

Irrational numbers exist. There are uncountably many real numbers.

König's lemma. Let G be a connected graph with infinitely many vertices, in which all vertex degrees are finite. Then, G contains an infinitely long path.

Sequential compactness. Every closed and bounded subset $K \subset \mathbb{R}^n$ has the following property: every sequence $x_1, x_2, \ldots$ of points from K contains a subsequence $x_{i_1}, x_{i_2}, \ldots$ that converges to a point in K.

Continuous function on compact set. If $f : \Omega \to \mathbb{R}$ is a continuous function from a compact domain Ω, then there is some $x \in \Omega$ such that $f(x) \geq f(y)$ for all $y \in \Omega$.

2 Problems

1. Let S be a closed subset of the plane, and suppose that its projection onto the x-axis is bounded. Prove that its projection onto the y-axis is closed.
2. What if the assumption of bounded-x-axis-projection is removed?
3. Let $f : [0, \pi] \to \mathbb{R}$ be a continuous function satisfying $\int_0^\pi f(x) \sin x dx = 0 = \int_0^\pi f(x) \cos x dx$. Show that $f(x) = 0$ for at least two distinct values of x in the open interval $0 < x < \pi$. (If you are stuck trying to show that there are at least two zeros, try to show that there is at least one zero first.)
4. Use the previous problem to show that for every convex polygon P in the plane, there are at least 3 lines ℓ such that P's centroid is on ℓ, and it is exactly halfway between the points where ℓ intersects the boundary of P.
5. A *compact* set of real numbers is one which is both closed and bounded. Show that there are compact sets $A_1, A_2, \ldots$ of real numbers such that every compact set $K \subseteq \mathbb{R}$ is contained in some A_n (it may be contained in many A_n).
6. Show that if we are given compact sets $A_1, A_2, \ldots$ of *rational* numbers, then there is always some compact set of rationals $K \subseteq \mathbb{Q}$ such that $K \not\subseteq A_n$ for all n.
7. Let K be a convex open set in the plane, and let $P \in K$. Suppose that no infinite ray from P is entirely contained in K. Prove that K must be bounded.
8. Does the previous result hold without the assumption that K is open?
9. Let f and g be continuous real functions with period 1. Prove that

$$\lim_{n \to \infty} \int_0^1 f(x) g(nx) dx = \left(\int_0^1 f(x) dx \right) \left(\int_0^1 g(x) dx \right).$$

10. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that for every real $\alpha > 0$, $\lim_{n\to\infty} f(n\alpha) = 0$. Prove that $\lim_{x\to\infty} f(x) = 0$.

11. A sequence $x_1, x_2, \ldots$ has a *Cesaro limit* iff

$$\lim_{n\to\infty} \frac{x_1 + \cdots + x_n}{n}$$

exists. Find all functions $f : [0, 1] \to \mathbb{R}$ with the property that:

$$\{x_1, x_2, \ldots \text{has a Cesaro limit}\} \iff \{f(x_1), f(x_2), \ldots \text{has a Cesaro limit}\}.$$

3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class.