12. Geometry

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1 Famous results

Fermat point. Let ABC be a triangle. If all angles are less than or equal to $120°$, then there is a point F such that the angles $\angle AFB$, $\angle BFC$, and $\angle CFA$ are all equal to $120°$, and F is the unique point which minimizes the sum of the distances $FA + FB + FC$.

Erdős-Szekeres. For every integer n, there is some finite N such that whenever there are N distinct points in the plane, no 3 collinear, some n of them are in convex position.

Reflection principle. The n-th Catalan number C_n is equal to the number of paths along the edges of a grid with $n \times n$ cells, from the southwest corner to the northeast corner, where each path only takes steps in the north and east directions, and never crosses above the main diagonal. It turns out that $C_n = \frac{1}{n+1}\binom{2n}{n}$.

2 Problems

1. Given two points P and Q on the same side of a line ℓ, find the point $X \in \ell$ which minimizes the sum of the distances $XP + XQ$.

2. Given two points P and Q on the same side of a line ℓ, find the point X which minimizes the sum of the distances from X to P, Q, and ℓ. To minimize cases, consider only the situation when ℓ is the x-axis, P has coordinates $(0, 3)$, and Q has coordinates $(5, 4)$. It's sufficient to characterize X by explaining how you would construct it; you don't need to work out its actual coordinates.

3. A convex polygon E lies in the closed unit square (its vertices are permitted to be on the boundary of the unit square). Prove that the sum of the squares of the side lengths of E is at most 4.

4. (Happy Ending Problem.) Suppose that P_1, P_2, P_3, P_4, P_5 are distinct points in $\mathbb{R}^2$, and no 3 of them are collinear. Prove that some 4 of them are in *convex position*, i.e., they form the corners of a convex polygon.

5. Suppose that there are 4 distinct points in the plane, no three of which are collinear, and such that the four points do not all lie on a common circle. Show that one point lies inside the circle through the other three.

6. Suppose that n points have been given in the plane, no 3 of which are collinear. Prove that they can be labeled $P_1, P_2, P_3, \ldots, P_n$ so that $P_1P_2 \ldots P_n$ is a simple closed polygon (the only intersections between edges are at the common endpoints of consecutive edges). Note that the polygon is not required to be convex.

7. An ellipsoid is a 3-dimensional body consisting of points of the form $\{(x, y, z) : (x/a)^2 + (y/b)^2 + (z/c)^2 = 1\}$, where a, b, c are the lengths of its semi-axes. If $a < b < c$, determine the radius of the largest circle which exists on the surface of the ellipsoid.

8. Prove that no matter how 3 points are placed in the closed unit square (the set of all (x, y) satisfying $0 \le x \le 1$ and $0 \le y \le 1$), some two of them are distance $\le \sqrt{6} - \sqrt{2}$ apart.

9. Suppose that a triangle in the plane has vertices which are all lattice points. Show that its circumcircle has diameter less than or equal to the product of its side lengths.

3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class.