10. Combinatorics

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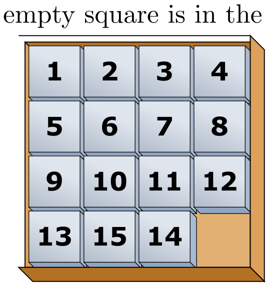
1 Famous results

Goodman. Suppose that each edge of the complete graph K_n has been colored either red or blue. The number of monochromatic triangles is always at least $(1 - o(1))\frac{n^3}{24}$, and this is asymptotically sharp.

Gallai, Hasse, Roy, Vitaver. Let D be a directed graph, and let χ be the chromatic number of its underlying undirected graph. Show that D has a directed path of at least χ vertices.

Fifteen puzzle. It is impossible to "solve" the following puzzle by shifting squares around until the 14 and 15 are in the correct order, and the empty square is in the lower right corner.



Sauer-Shelah. A family $\mathcal{F}$ shatters a set A if for every $B \subset A$, there is $F \in \mathcal{F}$ such that $F \cap A = B$. Prove that if $\mathcal{F} \subset 2^{[n]}$ and $|\mathcal{F}| > \binom{n}{0} + \cdots + \binom{n}{k}$, then there is a set $A \subset [n]$ of size $k+1$ such that $\mathcal{F}$ shatters A.

2 Problems

1. There are n players in a tournament. Each player plays every other player exactly once, and there are no ties. Let W_i be the number of games that player i won, and let L_i be the number of games that player i lost. Prove that

$$\sum_{i=1}^{n} W_i = \sum_{i=1}^{n} L_i \qquad \text{and} \qquad \sum_{i=1}^{n} W_i^2 = \sum_{i=1}^{n} L_i^2 .$$

2. There are n cities, and some pairs of them are linked by nonstop flights (if a pair is linked, there are flights going in both directions). For the i-th city, let d_i be the number of cities linked to i by flights. A 1-stop trip is a sequence of flights from city i to city j, and then from city j to city k (where k is allowed to equal i). Prove that the total number of possible 1-stop trips is exactly $\sum d_i^2$. (In graph-theoretic language, prove that the number of 2-edge walks on an undirected graph is always equal to the sum of the squares of the degrees.)

3. Suppose that $2n$ checkers have been placed on $n \times n$ chessboard. Prove that no matter how they have been placed, it is always possible to find a sequence of distinct checkers $C_1, C_2, \ldots, C_{2t}$ such that C_1 and C_2 are in the same row, C_2 and C_3 are in the same column, C_3 and C_4 are in the same row, C_4 and C_5 are in the same column, $\ldots$, C_{2t-1} and C_{2t} are in the same row, and C_{2t} and C_1 are in the same column. (Note that t can be less than n, i.e., it is not required to use all of the checkers in the sequence.)

4. In the year 201X, CMU entered 200 students in the Putnam. It turned out that every one of the 6 questions in the first half was fully solved by at least 120 CMU students. Show that there must then have been two CMU students so that each of the 6 problems was fully solved by at least one of them.

5. Let the *order* of a permutation σ be the minimum positive integer k so that σ^k (σ composed with itself a total of k times) is the identity permutation. Let $M(n)$ be the smallest positive integer t such that σ^t is the identity for every permutation σ of $\{1, 2, \ldots, n\}$. Prove that when n is not a prime power, then $M(n) = M(n-1)$, but when n is a prime power, say p^k, then $M(n) = pM(n-1)$.

6. Let $\mathcal{F}$ be a family of subsets of $\{1, 2, \ldots, n\}$, such that for every $A \in \mathcal{F}$, its complement $\{1, 2, \ldots, n\} \setminus A$ is also in $\mathcal{F}$, and for every $A, B \in \mathcal{F}$, both of $A \cup B$ and $A \cap B$ are also in $\mathcal{F}$. What are the possible sizes of $\mathcal{F}$?

7. Let $\mathcal{F}$ be a family of finite sets of natural numbers such that every pair $A, B \in \mathcal{F}$ has nonempty intersection $A \cap B \neq \emptyset$. Is there always a finite set U such that for every $A, B \in \mathcal{F}$, U has nonempty intersection with $A \cap B$, i.e., $U \cap A \cap B \neq \emptyset$? What if we further assume that all members of $\mathcal{F}$ have the same size?

3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class.