5. Functional equations

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1 Famous results

Cauchy. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function that satisfies $f(x+y) = f(x)+f(y)$ for all $x, y \in \mathbb{R}$. Then there must be a real number c such that $f(x) = cx$ for all $x \in \mathbb{R}$.

Cauchy-Schwarz. Let v and w be vectors in an inner product space. Then

$$|\langle v, w\rangle|^2 \leq \langle v, v\rangle \cdot \langle w, w\rangle \,,$$

with equality only if v and w are proportional.

Triple iterate. Let $f(x) = 1 - \frac{1}{x}$. Then $f(f(f(x))) = x$.

2 Problems

1. Let $X = \mathbb{R} \setminus \{0, 1\}$. Find all functions $f : X \to \mathbb{R}$ that satisfy

$$f(x) + f\left(1 - \frac{1}{x}\right) = 1 + x$$

for all $x \in X$.

2. Let α be a real number. Are there any continuous real-valued functions $f : [0, 1] \to \mathbb{R}^+$ such that

$$\int_0^1 f(x)dx = 1\,, \qquad \int_0^1 xf(x)dx = \alpha\,, \text{and} \qquad \int_0^1 x^2 f(x)dx = \alpha^2\,?$$

3. Let $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ be strictly monotone increasing, meaning that $f(x) < f(y)$ for all $x < y$. Suppose that $f(2) = 2$, and for every positive integers x, y with $\gcd(x, y) = 1$, we have $f(xy) = f(x)f(y)$. Prove that $f(x) = x$ for all x.

4. Find all continuously differentiable functions from $\mathbb{R} \to \mathbb{R}^+$, if any, which satisfy $f'(x) = f(x)$ for all x. Then, find all continuously differentiable functions from $\mathbb{R} \to \mathbb{R}^+$, if any, which satisfy $f'(x) = f(f(x))$ for all x. What if the range is allowed to be all of $\mathbb{R}$?

5. Find all twice differentiable functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy $f(x)^2 - f(y)^2 = f(x + y)f(x - y)$ for all $x, y \in \mathbb{R}$.