4. Calculus

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1 Famous results

Mean value theorem. For every function $f : [a, b] \to \mathbb{R}$ which is continuous on $[a, b]$ and differentiable in (a, b), there exists $\xi \in (a, b)$ such that

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}.$$

Cauchy's mean value theorem. For every functions $f, g : [a, b] \to \mathbb{R}$ which are continuous on $[a, b]$ and differentiable in (a, b), there exists $\xi \in (a, b)$ such that

$$(f(b) - f(a))g'(\xi) = (g(b) - g(a))f'(\xi).$$

If it happens that neither denominator is zero, then this is equivalent to the nicer expression:

$$\frac{f'(\xi)}{g'(\xi)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Taylor's theorem, with remainder. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and k-times-differentiable on (a, b), and let c be a real number in the interval (a, b). Then, for every $x \in (a, b)$, there there exists a ξ between c and x such that

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots + \frac{f^{(k-1)}(c)}{(k-1)!}(x-c)^{k-1} + \frac{f^{(k)}(\xi)}{k!}(x-c)^k.$$

Dirac delta "function". Physicists find the following "function" useful: $\delta(x) = 0$ for every real number x except $x = 0$, where $\delta(0) = \infty$, and

$$\int_{-\infty}^{\infty} \delta(x) dx = 1.$$

2 Problems

1. Determine

$$\lim_{n \to \infty} \sum_{i=1}^{n^2} \frac{n}{n^2 + i^2}.$$

2. Prove that

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2} = \frac{22}{7} - \pi.$$

3. Let $b \geq 2$ be a real number, and let $f : [0, b] \to \mathbb{R}$ be a twice differentiable function which satisfies $|f(x)| \leq 1$ and $|f''(x)| \leq 1$ for all $x \in [0, b]$. Prove that $|f'(x)| \leq 2$ for all $x \in [0, b]$.

4. Is there a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that for every $\epsilon > 0$,

$$\int_{-\epsilon}^{\epsilon} f(x)dx \geq 1?$$

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $\int_{-\infty}^{\infty} f(x)dx$ exists. Prove that

$$\int_{-\infty}^{\infty} f\left(x - \frac{1}{x}\right) dx = \int_{-\infty}^{\infty} f(x)dx$$

6. Prove that

$$\int_0^1 x^x dx = 1 - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + \cdots$$

7. Let $a \geq b \geq c > 0$ be real numbers. Define the random variables X, Y, Z as follows: let X be uniformly distributed on the interval $[0, a]$, let Y be uniform on $[0, b]$, and let Z be uniform on $[0, c]$. Calculate the expected value of $\min\{X, Y, Z\}$.

8. Let $f : \mathbb{R}^+ \to \mathbb{R}$ be twice continuously differentiable, with $\lim_{x\to 0^+} f'(x) = -\infty$ and $\lim_{x\to 0^+} f''(x) = +\infty$. Prove that

$$\lim_{x\to 0^+} \frac{f(x)}{f'(x)} = 0.$$

9. Let $\mathbb{D}$ be the closed unit disk $\{(x, y) : x^2 + y^2 \leq 1\}$ in $\mathbb{R}^2$, and suppose that $f : \mathbb{D} \to [-1, 1]$ is differentiable in $\mathbb{D}$, with respect to each of x and y. Prove that there is a point $(a, b) \in \mathbb{D}$ such that $f_x(a, b)^2 + f_y(a, b)^2 \leq 16$, where f_x and f_y denote the partial derivatives of f with respect to x and y.

3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class.