# 3. Number theory

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## 1 Famous results

**Fermat's Little Theorem.** For every prime $p$ and any integer $a$ which is not divisible by $p$, we have $a^{p-1} \equiv 1 \pmod{p}$.

**Euler.** Let $\varphi(n)$ denote the number of positive integers in $\{1, 2, \ldots, n\}$ which are relatively prime to $n$. Then, for any integer $a$ which is relatively prime to $n$,

$$a^{\varphi(n)} \equiv 1 \pmod{n}.$$

**Frobenius coin problem.** Suppose that a country has two types of coins, worth $a$ and $b$, where $a$ and $b$ are relatively prime. Then, the largest integer value which *cannot* be obtained through the coins is $ab - a - b$. However, if the country has three types of coins, worth $a$, $b$, and $c$, then there is no explicit formula known for the largest unattainable integer value.

## 2 Problems

1. Prove that for every integer $n > 1$, $n$ does not divide $2^n - 1$.

2. Suppose that an infinite arithmetic progression of positive integers contains a perfect $n$-th power (some $a^n$ for an integer $a$). Show that it must then contain infinitely many perfect $n$-th powers.

3. For positive integers $n$, define the function $f(n)$ as follows: write $n$ as a product of (not necessarily distinct) primes $p_1 p_2 \cdots p_t$, and let $f(n) = (-1)^t$. For example, $f(24) = (-1)^4$ because $24 = 2\times2\times2\times3$. Define
$$F(n) = \sum_{d|n} f(d).$$
Prove that for all positive integers $n$, the value of $F(n)$ is either 0 or 1, and characterize the $n$ for which $F(n) = 1$.

4. McDonalds sells Chicken McNuggets in boxes of size $a$ and $b$, where $a$ and $b$ are positive integers (of course). If you are hungry but picky, and would like to order exactly $n$ McNuggets, the only way to do this is to order some combination of full boxes of the two available sizes. A sharp kid observes that it is not possible to obtain exactly 58 McNuggets in this way, but that there are exactly 35 impossible integer values. Find $a$ and $b$.

5. Show that if a positive integer $n$ is a multiple of 24, then if one adds up all of the positive divisors of $n - 1$ (including 1 and $n - 1$), the total is also divisible by 24.

6. Suppose that for some real number $\alpha$, all of $1^\alpha, 2^\alpha, 3^\alpha, \ldots$ are integers. Prove that $\alpha$ is a nonnegative integer.

## 3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class.