2. Polynomials

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CMU Putnam Seminar, Fall 2013

1 Famous results

Single-variable. Suppose that the polynomial $P(z) = a_d z^d + a_{d-1} z^{d-1} + \cdots + a_0$ has $d+1$ distinct zeros. Then $P(z)$ is the zero polynomial, i.e., all $a_k = 0$. This works over any field.

Multi-variable. Let $P(x, y) = \sum_{i=0}^{d} \sum_{j=0}^{d} a_{i,j} x^i y^j$ be a polynomial, and let A_x, A_y be two (not necessarily distinct) sets of size $d+1$, such that $P(x, y) = 0$ for every $x \in A_x$, $y \in A_y$. Then $P(x, y)$ is the zero polynomial, i.e., all $a_{i_j} = 0$. This works over any field, and it generalizes to more than two variables.

Zero multiplicity. If a polynomial $p(z)$ has a root of multiplicity exactly m at $z = r$, then the $(m-1)$-st derivative of p at $z = r$ is 0, the m-th derivative is nonzero, and $p'(z)$ has a root of multiplicity exactly $m-1$ at $z = r$.

2 Problems

1. Find all real polynomials $p(z)$ with the following property: for every real polynomial $q(z)$, the two polynomials $p(q(z))$ and $q(p(z))$ are equal.

2. Find all polynomials $p(z)$ which satisfy both $p(0) = 0$ and $p(z^2 + 1) = p(z)^2 + 1$.

3. Let $p(z)$ be a degree-n polynomial over $\mathbb{C}$, with $n \geq 1$. Prove that there are at least $n+1$ distinct complex numbers $z \in \mathbb{C}$ for which $p(z) \in \{0, 1\}$.

4. (Binomial theorem for falling factorials.) For any positive integer n and any real number x, let the *falling factorial* $(x)_n$ be the product of n numbers $x(x-1)(x-2)\cdots(x-n+1)$. Prove that

$$(x+y)_n = \sum_{k=0}^{n} \binom{n}{k} (x)_k (y)_{n-k} .$$

   This also holds for rising factorials $x^{(n)} = x(x+1)\cdots(x+n-1)$.

5. A weather station measures the temperature T continuously. Meteorologists discover that every day, the temperature T follows some polynomial curve $p(t)$ with degree ≤ 3. (The particular polynomial may change from day to day.) Show that we can find times $t_1 < t_2$, which are independent of the polynomial p, such that the average temperature over the period 9am to 3pm is $\frac{1}{2}(p(t_1) + p(t_2))$, with $t_1 \approx$ 10:16am and $t_2 \approx$ 1:44pm.

6. Is there an infinite sequence $a_0, a_1, a_2, \ldots$ of nonzero real numbers such that for $n = 1, 2, 3, \ldots$ the polynomial

$$p_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

   has exactly n distinct real roots?

7. Let $p(z)$ be a degree-n polynomial with real coefficients, all of whose roots are real. Prove that

$$(n-1)p'(z)^2 \geq np(z)p''(z)$$

for all z, and determine all polynomials $p(z)$ for which

$$(n-1)p'(z)^2 = np(z)p''(z)\,.$$

3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class.