Putnam C.7

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12 October 2011

1 Problems

Putnam 1991/A1. A 2×3 rectangle has vertices at $(0,0)$, $(2,0)$, $(0,3)$, and $(2,3)$. It rotates 90° clockwise about the point $(2,0)$. It then rotates 90° clockwise about the point $(5,0)$, then 90° clockwise about the point $(7,0)$, and finally, 90° clockwise about the point $(10,0)$. (The side originally on the x-axis is now back on the x-axis.) Find the area of the region above the x-axis and below the curve traced out by the point whose initial position is $(1,1)$.

Putnam 1991/A2. Let $\mathbf{A}$ and $\mathbf{B}$ be different $n \times n$ matrices with real entries. If $\mathbf{A}^3 = \mathbf{B}^3$ and $\mathbf{A}^2\mathbf{B} = \mathbf{B}^2\mathbf{A}$, can $\mathbf{A}^2 + \mathbf{B}^2$ be invertible?

Putnam 1991/A3. Find all real polynomials $p(x)$ of degree $n \geq 2$ for which there exist real numbers $r_1 < r_2 < \cdots < r_n$ such that

1. $p(r_i) = 0$ for all $i = 1, 2, \ldots, n$, and
2. $p'\left(\frac{r_i + r_{i+1}}{2}\right) = 0$ for all $i = 1, 2, \ldots, n-1$,

where $p'(x)$ denotes the derivative of $p(x)$.