11. Geometry

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CMU Putnam Seminar, Fall 2011

1 Classical results

Triangle area. Let ABC be a triangle with side lengths $a = BC$, $b = CA$, and $c = AB$, and let r be its inradius and R be its circumradius. Let $s = \frac{a+b+c}{2}$ be its semiperimeter. Then its area is

$$sr = \sqrt{s(s-a)(s-b)(s-c)} = \frac{abc}{4R} = \frac{1}{2}ab\sin C\,.$$

Pick. The area of any polygon with integer vertex coordinates is exactly $I + \frac{B}{2} - 1$, where I is the number of lattice points in its interior, and B is the number of lattice points on its boundary.

Descartes. Let a, b, c, d be the radii of four mutually tangent circles. Let w, x, y, z be their inverses. Then

$$w^2 + x^2 + y^2 + z^2 = \frac{1}{2}(w + x + y + z)^2\,.$$

2 Problems

VTRMC 2001/2. Two circles with radii 1 and 2 are placed so that they are tangent to each other and a straight line. A third circle is nestled between them so that it is tangent to the first two circles and the line. Find the radius of the third circle.

VTRMC 1998/2. The radius of the base of a right circular cone is 1. The vertex of the cone is V, and P is a point on the circumference of the base. The length of PV is 6 and the midpoint of PV is M. A piece of string is attached to M and wound tightly twice round the cone finishing at P. What is the length of the string?

IMO 1999/1. Find all finite sets S of at least three points in the plane such that for all distinct points $A, B \in S$, the perpendicular bisector of AB is an axis of symmetry for S.

USAMO 1997/4. A sequence of polygons is derived as follows. The first polygon is a regular hexagon of area 1. Thereafter each polygon is derived from its predecessor by joining two adjacent edge midpoints and cutting off the corner. Show that all the polygons have area greater than 1/3.

VTRMC 2005/4. A cubical box with sides of length 7 has vertices at (0, 0, 0), (7, 0, 0), (0, 7, 0), (7, 7, 0), (0, 0, 7), (7, 0, 7), (0, 7, 7), (7, 7, 7). The inside of the box is lined with mirrors and from the point (0, 1, 2), a beam of light is directed to the point (1, 3, 4). The light then reflects repeatedly off the mirrors on the inside of the box. Determine how far the beam of light travels before it first returns to its starting point at (0, 1, 2).

Putnam 2000/A5. Three distinct points with integer coordinates lie in the plane on a circle of radius $r > 0$. Show that two of these points are separated by a distance of at least $r^{1/3}$.